INITIAL STATES and BRANCHING RATIOS in pp ANNIHILATIONS AT REST

U. Gastaldi, G. Maron, L. Vannucci, and G. Vedovato

I.N.F.N. Laboratori Nazionali di Legnaro - Legnaro (Pd) Italy

INITIAL STATES and BRANCHING RATIOS
in pp̄ ANNIHILATIONS AT REST

U. Gastaldi*, G. Maron* , L. Vannucci* and G. Vedovato*

INFN Laboratori Nazionali di Legnaro
Via Romea 4, I-35020 Legnaro (Padova)

Abstract

pp̄ annihilations at rest occur from initial states with J^{PC} = 0^{-+}. 1^{--}, 1^{+-}, 0^{++}, 1^{++}, 2^{++}. These initial states are S- and P-wave atomic levels of protonium atoms that are populated by the atomic cascade which follows the atomic capture of the antiproton. Each of the six fractions $F_{J^{PC}}$ of annihilations from initial states of protonium with quantum numbers J^{PC} depends from the density of the target where antiprotons are brought to rest, and is not known individually at present at any target density.

In this paper we show that the values of the 6 annihilation fractions $F_{J^{PC}}$ can be determined by measuring at four target densities the values $f_{J^{PC}}^{channel}$ of the fractions of annihilations for six exclusive annihilation channels. The same measurements would provide, for the channels exploited for the $F_{J^{PC}}$ determination, the values $B_{J^{PC}}^{channel}$ of the branching ratios of annihilation from each J^{PC} type of initial states from which the channels are allowed.

---

* Visitor at CERN, Geneva, Switzerland.

1. INTRODUCTION

$p\bar{p}$ annihilations at rest occur from the singlet 0^{-+} and the triplet 1^{--} S-wave atomic states of protonium with principal quantum number n higher or equal to 1, and from the singlet 1^{+-} and triplet 0^{++}, 1^{++} and 2^{++} P-wave hyperfine states with n higher or equal to 2. Annihilations do not occur from D-wave and higher orbital angular momentum states because of the negligible overlap of the p and $\bar{p}$ wave functions.

The populations of the levels of protonium and in particular of those from which annihilation takes place depend from the density ρ of the H_2 target where antiprotons are brought to rest. The distribution of the initial states of annihilation according to their J^{PC} value depends then from the target density.

Experimental data from LEAR show indeed that variations of the density of the target are accompanied by variations of the fractions of some exclusive channels of annihilation, as e.g. K^+K^-, K_SK_L, K_SK_S, $\pi^+\pi^-$ [1–3] by variations of the ratio between some of these fractions [1–4], and by variations of the absolute and relative yields of radiative transitions between protonium levels [5–11].

These fenomena are usually discussed within a symplified framework where only two types of annihilation are considererd: S-wave and P-wave annihilations. The ASTERIX experiment deduced within this framework the fraction of S-wave and of P-wave annihilations in a H_2 target at NTP and in a liquid H_2 target and the values of the branching ratios of some exclusive final states for what was called pure S-wave annihilation and pure P-wave annihilation [1,2]. The experimental inputs were:

i) the branching fractions $^{P}f^{channel}$ measured by ASTERIX in a H_2 gas target at NTP for the two body final states $\pi^+\pi^-$, K^+K^-, K_SK_S and K_SK_L,

ii) the branching fractions $^{x}f^{channel}$ measured by ASTERIX for the two body final states $\pi^+\pi^-$, K^+K^-, K_SK_S and K_SK_L, produced in coincidence with a X-ray in the 1–4 keV energy window in the same H_2 gas target at NTP,

iii) the ratio between the yield of protonium L X-rays and the yield of background X-rays due to internal bremstrahlung and fluorescence caused by the charged mesons produced by the $p\bar{p}$ annihilation or by decays of neutral mesons,

iv) the branching fractions $^{L}f^{channel}$ measured in liquid H_2 targets for the channels $\pi^+\pi^-$, K^+K^-, K_SK_L and $\pi^0\pi^0$.

The ASTERIX analysis was based on the simpificatory schematization of annihilation in terms of only two components and relied on tacitly assumed hypothesis [12] of which the most critical one was the assumption that the fractions of annihilations from the four P-wave sublevels with principal quantum number n=2 were proportional to the four $F_{J^{PC}}$ fractions of annihilation for P-wave annihilations at the same target density. Within the context of the symplifications mentioned above the ASTERIX analysis permitted to determine a significant value for the ratio between S-wave and P-wave annihilations in H_2 gas at NTP and a value of that ratio in liquid H_2 that was much more uncertain, because it depended critically from the magnitude of the fraction $^{Liquid}f^{\pi^0\pi^0}$, whose experimental values varied from (4.8 to 1.4 to 2.06 to 2.5) 10^{-4} [13–16] and has grown recently to 6.9×10^{-4} [17]. The derivation of the fraction of P-wave annihilation in H_2 gas at 15 atm was obtained by CPLEAR [4] following the track of the ASTERIX analysis. The values derived for the ratio between S- and P-wave annihilations in gas and in liquid are exploited in quantitative analyses of $p\bar{p}$ annihilations at rest with three and more mesons produced in the final state and with conventional and exotic mesons in the intermediate states [18–23] and in the determination of the electromagnetic form factor of the proton at threshold in the timelike region [24,25].

In this paper we express the structure of the J^{PC} distribution of the initial states of $p\bar{p}$ annihilaton at rest in terms of atomic states of protonium. We show that the six annihilation fractions $F_{J^{PC}}$, which are defined in Section 2.4, can be determined by measuring the fractions of some exclusive annihilation channels and the angular distributions of the particles measured in the final state. This task requires measurements at several target densities and will simultaneously yild the branching ratios for annihilation from each allowed J^{PC} initial state for the channels exploited. The method that we suggest to derive the annihilation fractions and the branching ratios can be completely independent from any hypotesis about the atomic cascade and the effects of strong interactions on the atomic levels of protonium. It can be simplified by using explicitly one dynamical hypotesis about the evolution of the atomic cascade. We show also that the main hypotesis exploited implicitly in the derivation of the S-versus P-wave annihilation ratios by ASTERIX is not justified.

This work elaborates on points anticipated in ref [26] and discussed [27] at the Nucleon-Antinucleon conference NAN '91. The additional physics information that can be extracted by performing a similar program in coincidence with detection of L and K X-rays of protonium is discussed elsewhere [28].

The determination of the annihilation fractions is necessary for a precise measurement of the electromagnetic form factor of the proton at the threshold of the timelike region [29] and for quantitative differential measurements in the spectroscopy of light mesons and exotics [30,31]. The determination of the branching ratios of annihilation channels from the different J^{PC} initial state from which they are allowed is a crucial experimental input for the study of the dynamics of nucleon antinucleon annihilations. Refs. [26,32–34] give experimental and theoretical reviews about this topic. The approach outlined in the present paper does not require to enter deeply into the details of the atomic cascade of protonium. The interested reader is therefore referred to the papers where the main mechanisms which make the atomic cascade of protonium density dependent were discussed [35–37], to recent reviews of protonium physics [38–40] and cascade calculations [41].

2. PROTONIUM LEVELS, CASCADE and ANNIHILATION
2.1 General points

Each antiproton brought at rest in a H_2 target forms a $p\bar{p}$ atom in a highly exited state. The $p\bar{p}$ atom undergoes its atomic cascade, wich finishes with annihilation in a S- or P-wave level of protonium. Protonium is analogous to positronium, but has a much larger reduced mass, leading to a Rydberg energy of 12.5 keV, and strong interactions perturbing considerably its S- and P-vave levels. In the ground state, that has a lifetime of about 10^{-18} sec, the separation between p and $\bar{p}$ is about 50 fm and grows as n^2 with increasing principal quantum number n. Annihilations occur only from S and P-states because for states with higher orbital angular momentum the overlap of the p and $\bar{p}$ wave functions is negligible.

For the same reason annihilation is much more strong in S-waves than in P-waves. The values of the shift ΔE and broadening Γ caused by strong interactions to the protonium levels are expected to vary with the principal quantum number n according to the overlap of the wave functions and to depend from the spin and the angular momentum (see Ref. 34 and references quoted therin):

$$\Gamma_{nJ^{PC}} = f(n)\ \Gamma_{J^{PC}}$$
$$\Delta E_{nJ^{PC}} = f(n)\ \Delta E_{J^{PC}} \tag{1}$$

where for S-states $f(n) = n^{-3}$ with $\Gamma_{J^{PC}} = \Gamma_{n=1,J^{PC}}$, and for P-states $f(n) = 32/3(n^{-3} - n^{-5})$ with $\Gamma_{J^{PC}} = \Gamma_{n=2,J^{PC}}$.

None of the 12 independent quantities $\Delta E_{J^{PC}}$ and $\Gamma_{J^{PC}}$, that caracterize the global effects of strong interactions on the two n=1 S-levels and the four n=2 P-levels of protonium has been measured directly so far. This implies that there are 12 parameters, that require experimental determination, which are a necessary input to a complete calculation of the cascade of protonium. Figure 2 and Table 1 in Ref. [38] show the scheme of the lower levels of protonium and the expected perturbations to their coulomb binding energies. The values reported there are those calculated in the model of Ref. [42] and are used just to illustrate the orders of magnitude and to stress that the values of ΔE and of Γ are expected to depend from the nJ^{PC} quantum numbers of the S- and P-levels. From experiments measuring X-rays of protonium [6-11], are available average values for ΔE and Γ of the ground state (which correspond to different experimental conditions in which the averages have been made), and an average width Γ for the n=2 P states.

S and P-wave levels of protonium that annihilate are completely identifed by the nJ^{PC} quantum numbers. The population of the levels of protonium from which annihilation may occur is indicated here as $^{P}P_{nJ^{PC}}$.

$^{P}P_{nJ^{PC}}$ represents the probability that the level be reached in the course of the atomic cascade. The population of a level depends from the atomic cascade and from the distribution of initial states of protonium at capture. For any level this population depends on the density ρ of the target, to which is proportional the number of collisions per unit time of protonium atoms with the molecules of the surrounding medium. The precise calculation of $^{P}P_{nJ^{PC}}$ is not possible at present because the 12 strong interaction values that caracterize the strong interaction effects on the levels of protonium are not individually known. For the purpose of this paper it is however sufficient to indicate explicitly the fact that the values of $^{P}P_{nJ^{PC}}$ are density dependent.

Any nJ^{PC} level is depopulated by two types of transitions. The first type of transitions are:

i) radiative transitions to lower lying levels of protonium with a decay width $\Gamma^{rad}_{nJ^{PC}}$ and

ii) annihilations with a decay width $\Gamma^{ann}_{nJ^{PC}}$.

The widths associated to these transitions are controlled only by the internal dynamics of protonium.

The second type of transitions include:

i) Chemical transitions (in the highly exited states of protonium) where the antiproton can change of proton partner during a collision,

ii) Auger transitions, where protonium makes transitions to lower lying levels transferring energy to electrons of molecules crossed during collisions,

iii) Stark transitions induced by mixing with nearby levels of equal n; the stark mixing is caused by the non-central electric field experienced by protonium when crossing Hydrogen molecules in the target.

The second type of transitions have rates proportional to the density of the target.

The total width of a protonium level is then density dependent and can be written as

$$^{\rho}\Gamma^{tot}_{nJ^{PC}} = \Gamma^{rad}_{nJ^{PC}} + \Gamma^{ann}_{nJ^{PC}} + {}^{\rho}\Gamma^{stark}_{nJ^{PC}} + {}^{\rho}\Gamma^{auger}_{nJ^{PC}} + {}^{\rho}\Gamma^{chem}_{nJ^{PC}} \qquad (2)$$

2.2 Definition of branching ratio of an annihilation channel from a level of given J^{PC}

We indicate with $\Gamma^{ch}_{nJ^{PC}}$ the width for annihilation into the specific channel ch of the nJ^{PC} level. $\Gamma^{ch}_{nJ^{PC}}$ obeys the same scaling law (1) as the total annihilation width $\Gamma^{ann}_{nJ^{PC}}$ of the level nJ^{PC}:

$$\Gamma^{ch}_{nJ^{PC}} = f(n) \ \Gamma^{ch}_{J^{PC}} \tag{3}$$

For a given exclusive annihilation channel ch we call $B^{ch}_{J^{PC}}$ the ratio between the probability of annihilation of a level nJ^{PC} in a channel ch $\Gamma^{ch}_{nJ^{PC}}$ and the total probability of annihilation $\Gamma^{ann}_{nJ^{PC}}$ of the level nJ^{PC}:

$$B^{ch}_{J^{PC}} = \Gamma^{ch}_{nJ^{PC}} / \Gamma^{ann}_{nJ^{PC}} = \Gamma^{ch}_{J^{PC}} / \Gamma^{ann}_{J^{PC}} \tag{4}$$

$B^{ch}_{J^{PC}}$ is the branching ratio for annihilations into the channel ch. This quantity is the same for all levels with the same J^{PC}, since it is independent of n and of the atomic cascade. It tells how much strong interactions favour annihilation into the channel ch for a protonium level of quantum numbers J^{PC}.

2.3 Fractions of annihilation from individual atomic levels

The contribution $^{\rho}f_{nJ^{PC}}$ of the nJ^{PC} level to annihilation is proportional to its population and to the fraction of its disappearance by annihilation over its total disappearance rate:

$$^{\rho}f_{nJ^{PC}} = {}^{\rho}P_{nJ^{PC}} \ \Gamma^{ann}_{nJ^{PC}} / {}^{\rho}\Gamma^{tot}_{nJ^{PC}} \tag{5}$$

The contribution $^{\rho}f^{ch}_{nJ^{PC}}$ of the nJ^{PC} level to annihilation in a specific channel ch is proportional to its population and to the fraction of its disappearance by annihilation into that channel ch over its total disappearence rate:

$$^{\rho}f^{ch}_{nJ^{PC}} = {}^{\rho}P_{nJ^{PC}} \ \Gamma^{ch}_{nJ^{PC}} / {}^{\rho}\Gamma^{tot}_{nJ^{PC}} \tag{6}$$

2.4 Annihilation fractions

We can reexpress (6) by using (4) in the following way:

$$^{\rho}f^{ch}_{nJ^{PC}} = B^{ch}_{J^{PC}} \ {}^{\rho}P_{nJ^{PC}} \ \Gamma^{ann}_{nJ^{PC}} / {}^{\rho}\Gamma^{tot}_{nJ^{PC}} \tag{7}$$

which says that the contribution of level nJ^{PC} in a target of density ρ to annihilation into the channel ch is proportional to the branching ratio of the channel ch from the J^{PC} level times the probability of populating the level nJ^{PC} times the probability that the level nJ^{PC} undergoes annihilation.

We can then reexpress (7),by using (5) as:

$$^\rho f^{ch}_{nJ^{PC}} = B^{ch}_{J^{PC}} \; {}^\rho f_{nJ^{PC}} \tag{8}$$

The fraction $^\rho F_{J^{PC}}$ of all annihilations from all the initial states of protonium of a given J^{PC} is given by the sum over n of the contributions $^\rho f_{nJ^{PC}}$ to annihilation of all the states nJ^{PC}:

$$^\rho F_{J^{PC}} = \sum_n {}^\rho f_{nJ^{PC}} = \sum_n {}^\rho P_{nJ^{PC}} \; \Gamma^{ann}_{nJ^{PC}} \; / \; {}^\rho \Gamma^{tot}_{nJ^{PC}} \tag{9}$$

Each annihilation fraction depends on the density of the target and on the 12 strong interaction parameters $\Delta E_{J^{PC}}$ and $\Gamma^{ann}_{J^{PC}}$, via the dependence of the terms $^\rho P_{nJ^{PC}}$, $\Gamma^{ann}_{nJ^{PC}}$ and $^\rho \Gamma^{tot}_{nJ^{PC}}$. Even if the dependence from the 12 strong interaction terms via the above first and third terms would be only mild, the dependence on the annihilation width $\Gamma^{ann}_{J^{PC}}$ is strong, as it can be seen rewriting (9) as:

$$^\rho F_{J^{PC}} = \Gamma^{ann}_{J^{PC}} \sum_n f(n) \; {}^\rho P_{nJ^{PC}} / {}^\rho \Gamma^{tot}_{nJ^{PC}} \tag{10}$$

The sum of the contributions of all the n levels of the same J^{PC} to annihilations in a given channel ch gives the fraction $^\rho f^{ch}_{nJ^{PC}}$ of annihilations for that channel from all protonium levels with that J^{PC}.
We can then write using (8),

$$^\rho f^{ch}_{J^{PC}} = \sum_n {}^\rho f^{ch}_{nJ^{PC}} = \sum_n B^{ch}_{J^{PC}} \; {}^\rho f_{nJ^{PC}} = $$

$$= B^{ch}_{J^{PC}} \; {}^\rho F_{J^{PC}} \tag{11}$$

2.5 Experimental determination of fractions of annihilation and of partial fractions of annihilation into exclusive channels

In a target of density ρ the fraction $^\rho f^{ch}$ of annihilations into an exclusive channel ch is given by the ratio between the number $^\rho N^{ch}$ of annihilations into the channel ch and the number $N_{\bar{p}}$ of antiprotons annihilating into the target:

$$^\rho f^{ch} = {}^\rho N^{ch} \; / \; N_{\bar{p}} \tag{12}$$

If the channel ch can be produced from more than one J^{PC}, we can write $^\rho N^{ch}$ as

$$^\rho N^{ch} = \Sigma_{J^{PC}} \; {}^\rho N^{ch}_{J^{PC}} \tag{13}$$

If the analysis of the decay angular distributions permits to determine the individual contributions $^\rho N^{ch}_{J^{PC}}$ to $^\rho N^{ch}$ it is than possible to determine experimentally the partial fractions, that are given by the ratio between the number $^\rho N^{ch}_{J^{PC}}$ of annihilations into the

channel ch from all the states with quantum number J^{PC} and the number $N_{\bar{p}}$ of antiprotons annihilating into the target:

$$^{\rho}f_{J^{PC}}^{ch} = {}^{\rho}N_{J^{PC}}^{ch} / N_{\bar{p}} \tag{14}$$

3. DETERMINATION OF ANNIHILATION FRACTIONS AND OF BRANCHING RATIOS

3.1 General points

Equations 14 and 12 express in terms of experimental observables the same quantities described in terms of sums of the contributions of the individual atomic levels respectively by equations 8 and 11. We have then

$$\Sigma_{J^{PC}} \quad B_{J^{PC}}^{ch} \quad {}^{\rho}F_{J^{PC}} \; = \; {}^{\rho}f^{ch} \; = \; {}^{\rho}N^{ch} / N_{\bar{p}} \tag{15}$$

$$B_{J^{PC}}^{ch} \quad {}^{\rho}F_{J^{PC}} \; = \; {}^{\rho}f_{J^{PC}}^{ch} \; = \; {}^{\rho}N_{J^{PC}}^{ch} / N_{\bar{p}} \tag{16}$$

Equation 16 tells us that, if the annihilation fraction $^{\rho}F_{J^{PC}}$ at a given target density ρ is known for the levels with discrete quantum numbers J^{PC}, then the branching ratio $B_{J^{PC}}^{ch}$ for annihilation into a channel ch from the levels with discrete quantum numbers J^{PC} is derivable directly from the measurement of the partial fraction of annihilation $^{\rho}f_{J^{PC}}^{ch}$ of that channel. The same measurement at a second target density permits to check the value of the branching ratio, if the annihilation fraction at this second density is already known, or conversely to determine the annihilation fraction, by using the branching ratio value given by the first measurement.

Table 1 explicitates conditions 15 and 16 for a number of $p\bar{p}$ annihilations into simple channels, that can be identified unambiguously and reconstructed with little background. Selection rules limit, sometimes drastically, the $p\bar{p}$ initial states that can contribute to annihilations into the channels listed (see Ref. 3 and references therein). Each of the channels listed in the table has already been observed experimentally at a target density or another in at least one exclusive final state [43]. For each J^{PC} it is possible to find one or more channels that are not forbidden by selection rules and whose $^{\rho}f_{J^{PC}}^{ch}$ be sizeable so that equation 16 can be exploited sensibly.

Since each antiproton brought to rest annihilates, the following relation links the values of the six annihilation fractions :

$$^{\rho}F_{0^{-+}} + {}^{\rho}F_{1^{--}} + {}^{\rho}F_{1^{+-}} + {}^{\rho}F_{0^{++}} + {}^{\rho}F_{1^{++}} + {}^{\rho}F_{2^{++}} = 1 \tag{17}$$

Since in the course of the atomic cascade none of the effects that contribute to the cascade acts directly on the total spin of protonium [44], atoms formed in singlet state will annihilate sooner or later in a singlet state, and analogously for atoms formed in triplet states. For a unpolarized $\bar{p}$ beam stopped in a unpolarized H_2 target, atomic capture will populate statistically the spin levels and equation 17 breaks down into the two separate equations:

$$^{\rho}F_{0^{-+}} + {}^{\rho}F_{1^{+-}} = 1 / 4 \tag{18}$$

$$^{\rho}F_{1^{--}} + {}^{\rho}F_{0^{++}} + {}^{\rho}F_{1^{++}} + {}^{\rho}F_{2^{++}} = 3 / 4 \tag{19}$$

We have written equations 18 and 19 under the explicit assumption about the dynamics of the protonium cascade that no effect acts directly on the total spin of the $p\bar{p}$ atom. Should this hypothesis be not justified, one can make only use of Eq. (17), which is quite general since the antiproton is a stable particle, and we consider a pure H_2 target (hence no atomic transfers to nuclei different from the proton). For the time being we assume the validity of equation 18 and 19, and we will discuss later on how to check experimentally a posteriory this assumption. Another implication of conditions 18 and 19 is that, if we consider S-wave annihilations only in a target of given density, the ratio between S-wave spin singlet and S-wave spin triplet annihilations needs not to be 1/3. If that ratio would be 1/3 at that given target density it will very likely not be 1/3 at a different density. The same holds if one would consider P-wave annihilations only.

3.2 Determination of ${}^P F_{0^{-+}}$ and of ${}^P F_{1^{+-}}$

Let us select within table 1 two final states that can be produced from $p\bar{p}$ 0^{-+} and 1^{+-} initial states, and whose ${}^P f_{J^{PC}}^{ch}$ can be measured separately at two target densities exploiting the decay angular distributions. We will have, for example, at the two densities 1 and 2

$$
\begin{aligned}
B_{0^{-+}}^{\pi\phi} \; {}^1F_{0^{-+}} &= {}^1f_{0^{-+}}^{\pi\phi} \\
B_{0^{-+}}^{\pi\phi} \; {}^2F_{0^{-+}} &= {}^2f_{0^{-+}}^{\pi\phi} \\
B_{1^{+-}}^{\pi\phi} \; {}^1F_{1^{+-}} &= {}^1f_{1^{+-}}^{\pi\phi} \\
B_{1^{+-}}^{\pi\phi} \; {}^2F_{1^{+-}} &= {}^2f_{1^{+-}}^{\pi\phi}
\end{aligned}
\tag{20}
$$

from which we can express the four unknown annihilation fractions ${}^1F_{0^{-+}}$, ${}^1F_{1^{+-}}$, ${}^2F_{0^{-+}}$ and ${}^2F_{1^{+-}}$ in terms of the four measured fractions of annihilations and of the two branching ratios $B_{0^{-+}}^{\pi\phi}$ and $B_{1^{+-}}^{\pi\phi}$.

If we relabel $B_{0^{-+}}^{\pi\phi} = 1/x$ and $B_{1^{+-}}^{\pi\phi} = 1/y$, the annihilation fractions can be expressed, using Eq. (20) in the following way:

$$
\begin{aligned}
{}^1F_{0^{-+}} &= x \; {}^1f_{0^{-+}}^{\pi\phi} \\
{}^2F_{0^{-+}} &= x \; {}^2f_{0^{-+}}^{\pi\phi} \\
{}^1F_{1^{+-}} &= y \; {}^1f_{1^{+-}}^{\pi\phi} \\
{}^2F_{1^{+-}} &= y \; {}^2f_{1^{+-}}^{\pi\phi}
\end{aligned}
\tag{21}
$$

We can now write down Eq. (18) at the two densities 1 and 2, and we get

$$
\begin{aligned}
x \; {}^1f_{0^{-+}}^{\pi\phi} + y \; {}^1f_{1^{+-}}^{\pi\phi} &= 1/4 \\
x \; {}^2f_{0^{-+}}^{\pi\phi} + y \; {}^2f_{1^{+-}}^{\pi\phi} &= 1/4
\end{aligned}
\tag{22}
$$

By solving this system of two linear equations with the two unknown x and y, one gets the two branching ratios

$$\frac{1}{x} = B^{\pi^0\phi}_{0^{-+}} = 4\ \frac{{}^2f^{\pi\phi}_{1^{+-}}\ {}^1f^{\pi\phi}_{0^{-+}} - {}^1f^{\pi\phi}_{1^{+-}}\ {}^2f^{\pi\phi}_{0^{-+}}}{{}^1f^{\pi\phi}_{1^{+-}} - {}^2f^{\pi\phi}_{1^{+-}}}$$

$$\frac{1}{y} = B^{\pi^0\phi}_{1^{+-}} = 4\ \frac{{}^2f^{\pi\phi}_{1^{+-}}\ {}^1f^{\pi\phi}_{0^{-+}} - {}^1f^{\pi\phi}_{1^{+-}}\ {}^2f^{\pi\phi}_{0^{-+}}}{-{}^1f^{\pi\phi}_{0^{-+}} + {}^2f^{\pi\phi}_{0^{-+}}}$$

$$(23)$$

By using the values of the two branching ratios in the expression 23 above, the two spin singlet annihilation fractions can be derived for each of the densities 1 and 2.

Cross check of these values can be obtained by using other annihilation channels at the same densities 1 and 2. The procedure not only allows the cross check, but gives the $B^{ch}_{J^{PC}}$ for the new channels used.

3.3 Determination of ${}^PF_{1^{--}}$, ${}^PF_{0^{++}}$, ${}^PF_{1^{++}}$ and ${}^PF_{2^{++}}$

Let us select within table 1 final states that can be produced from $p\bar{p}$ 1^{--}, 0^{++}, 1^{++} and 2^{++} initial states, and whose ${}^Pf^{ch}_{J^{PC}}$ can be measured conveniently at four target densities. We will have, for example, the following set of 16 measured fractions at the densities 1,2,3 and 4:

$$B^{K_sK_L}_{1^{--}}\ {}^1F_{1^{--}} = {}^1f^{K_sK_L}_{1^{--}}$$
$$B^{K_sK_L}_{1^{--}}\ {}^2F_{1^{--}} = {}^2f^{K_sK_L}_{1^{--}}$$
$$B^{K_sK_L}_{1^{--}}\ {}^3F_{1^{--}} = {}^3f^{K_sK_L}_{1^{--}}$$
$$B^{K_sK_L}_{1^{--}}\ {}^4F_{1^{--}} = {}^4f^{K_sK_L}_{1^{--}}$$

$$B^{\omega\phi}_{0^{++}}\ {}^1F_{0^{++}} = {}^1f^{\omega\phi}_{0^{++}}$$
$$B^{\omega\phi}_{0^{++}}\ {}^2F_{0^{++}} = {}^3f^{\omega\phi}_{0^{++}}$$
$$B^{\omega\phi}_{0^{++}}\ {}^3F_{0^{++}} = {}^3f^{\omega\phi}_{0^{++}}$$
$$B^{\omega\phi}_{0^{++}}\ {}^4F_{0^{++}} = {}^4f^{\omega\phi}_{0^{++}}$$

$$B^{\omega\phi}_{1^{++}}\ {}^1F_{1^{++}} = {}^1f^{\omega\phi}_{1^{++}}$$
$$B^{\omega\phi}_{1^{++}}\ {}^2F_{1^{++}} = {}^2f^{\omega\phi}_{1^{++}}$$
$$B^{\omega\phi}_{1^{++}}\ {}^3F_{1^{++}} = {}^3f^{\omega\phi}_{1^{++}}$$
$$B^{\omega\phi}_{1^{++}}\ {}^4F_{1^{++}} = {}^4f^{\omega\phi}_{1^{++}}$$

$$B^{\omega\phi}_{2^{++}}\ {}^1F_{2^{++}} = {}^1f^{\omega\phi}_{2^{++}}$$
$$B^{\omega\phi}_{2^{++}}\ {}^2F_{2^{++}} = {}^2f^{\omega\phi}_{2^{++}}$$
$$B^{\omega\phi}_{2^{++}}\ {}^3F_{2^{++}} = {}^3f^{\omega\phi}_{2^{++}}$$
$$B^{\omega\phi}_{2^{++}}\ {}^4F_{2^{++}} = {}^4f^{\omega\phi}_{2^{++}}$$

$$(24)$$

from which we can express the 16 unknown annihilation fractions in terms of the 16 measured fractions of annihilations and of the four branching ratios $B^{K_S K_L}$, $B^{\omega\phi}_{0^{++}}$, $B^{\omega\phi}_{1^{++}}$ and $B^{\omega\phi}_{2^{++}}$.

If we relabel $B^{K_S K_L}_{1^{--}} = 1/a$, $B^{\omega\phi}_{0^{++}} = 1/b$, $B^{\omega\phi}_{1^{++}} = 1/c$, $B^{\omega\phi}_{2^{++}} = 1/d$ and we write down equation 19 at the four densities exploited, we get the following system of four linear equations in the four unknowns a,b,c and d:

$$a \, {}^1f^{K_S K_L}_{1^{--}} + b \, {}^1f^{\omega\phi}_{0^{++}} + c \, {}^1f^{\omega\phi}_{1^{++}} + d \, {}^1f^{\omega\phi}_{2^{++}} = 3/4$$

$$a \, {}^2f^{K_S K_L}_{1^{--}} + b \, {}^2f^{\omega\phi}_{0^{++}} + c \, {}^2f^{\omega\phi}_{1^{++}} + d \, {}^2f^{\omega\phi}_{2^{++}} = 3/4$$

$$a \, {}^3f^{K_S K_L}_{1^{--}} + b \, {}^3f^{\omega\phi}_{0^{++}} + c \, {}^3f^{\omega\phi}_{1^{++}} + d \, {}^3f^{\omega\phi}_{2^{++}} = 3/4$$

$$a \, {}^4f^{K_S K_L}_{1^{--}} + b \, {}^4f^{\omega\phi}_{0^{++}} + c \, {}^4f^{\omega\phi}_{1^{++}} + d \, {}^4f^{\omega\phi}_{2^{++}} = 3/4 \qquad (25)$$

By solving this system of four linear equations with the four unknowns a, b, c and d one gets the values of the four branching ratios $B^{K_S K_L}_{1^{--}}$ $B^{\omega\phi}_{0^{++}}$, $B^{\omega\phi}_{1^{++}}$ and $B^{\omega\phi}_{2^{++}}$ in terms of the 16 frequencies measured.

By inserting the values of the branching ratios in the Eq. (24) above, the four spin triplet annihilation fractions can be derived for each of the densities 1, 2, 3 and 4.

Again, cross checks of these values can be obtained by using other annihilation channels at the same four densities . The procedure not only allows the cross check, but gives the BR for the new channels used.

If the cross checks were not to be satisfactory, then one should question the dynamical assumption that has permitted to break down Eq. (17) into Eq. (18) and (19). Determining branching ratios and annihilation fractions would then require to express the 6 annihilation fractions ${}^\rho F_{J^{PC}}$ in terms of 6 different branching ratios $B^{ch(i)}_{J^{PC}}$, to measure at 6 different densities identified by a label $d = 1 \rightarrow 6$, the 6 fractions of annihilations ${}^d f^{ch(i)}_{J^{PC}}$ associated to the 6 chosen branching ratios, to write equation 17 at the 6 different densities d, to get a system of 6 linear equations in the 6 unknowns corresponding to the inverse of the 6 branching ratios exploited, to solve that system in order to get the 6 values of the branching ratios selected, and finally to use these values of the branching ratios in order to get the 6 annihilation fractions for each of the 6 target densities employed.

4. COMMENTS
4.1 S-wave and P-wave annihilation

Given a target at a density ρ, the fractions ${}^\rho F_S$ and ${}^\rho F_P$ of S-wave and P-wave annihilations and the fractions ${}^\rho f^{ch}_S$ and ${}^\rho f^{ch}_P$ of S-wave and P-wave annihilations into a given channel ch are given respectively by

$$^\rho F_S = \quad {}^\rho F_{0^{-+}} \quad + {}^\rho F_{1^{--}} \qquad (26)$$

$$^\rho f^{ch}_S = B^{ch}_{0^{-+}} \, {}^\rho F_{0^{-+}} + B^{ch}_{1^{--}} \, {}^\rho F_{1^{--}} \qquad (27)$$

$$^\rho F_P = \quad {}^\rho F_{1^{+-}} \quad + {}^\rho F_{0^{++}} \quad + {}^\rho F_{1^{++}} \quad + {}^\rho F_{2^{++}} \qquad (28)$$

$$^\rho f^{ch}_P = B^{ch}_{1^{+-}} \, {}^\rho F_{1^{+-}} + B^{ch}_{0^{++}} \, {}^\rho F_{0^{++}} + B^{ch}_{1^{++}} \, {}^\rho F_{1^{++}} + B^{ch}_{2^{++}} \, {}^\rho F_{2^{++}} \qquad (29)$$

Comparison of expressions 26 and 27 and of expressions 28 an 29 shows immediately that in general ${}^{P}F_s$ is not proportional to ${}^{P}f_s^{ch}$ and ${}^{P}F_p$ is not proportional to ${}^{P}f_p^{ch}$. This could only occur for a channel allowed from all S or P-wave quantum numbers and with the same branching ratio from all quantum numbers.

4.2 Annihilations from the n=2 levels of protonium

The n=2 levels of protonium are populated dominantly by radiative transitions from levels with higher n. These radiative transitions can be detected in coincidence with the products of annihilation from the n=2 levels (or of annihilation from n=1 levels populated by a radiative transition from n=2 to n=1).

The fractions of annihilation from the n=2 levels of protonium can be expressed by specialising equation 5 :

$$ {}^{P}f_{2J^{PC}} = {}^{P}P_{2J^{PC}} \quad \Gamma^{ann}_{2J^{PC}}/{}^{P}\Gamma^{tot}_{2J^{PC}} \tag{30} $$

Let us reorder the terms in Eq. 30 and rewrite it together with Eq. 10

$$ {}^{P}f_{2J^{PC}} = \Gamma^{ann}_{J^{PC}} \quad f(2){}^{P}P_{2J^{PC}} \; / \; {}^{P}\Gamma^{tot}_{2J^{PC}} \tag{30} $$

$$ {}^{P}F_{J^{PC}} = \Gamma^{ann}_{J^{PC}} \sum_{n} f(n){}^{P}P_{nJ^{PC}} \; / \; {}^{P}\Gamma^{tot}_{nJ^{PC}} \tag{10} $$

Equations 10 and 30 and their comparison permits to extract a number of conclusions which are of general nature and not dependent from any assumptions about the evolution of the atomic cascade.

1) The fractions of annihilation in P-wave from levels n=2 of protonium depend from the target density.

2) The ratios between the four fractions of annihilation in P-wave from levels n=2 of protonium depend from the target density.

3) the ratios ${}^{P}f_{2J^{PC}} / {}^{P}F_{J^{PC}}$ depend from J^{PC}.

4) the ratios ${}^{P}f_{2J^{PC}} / {}^{P}F_{J^{PC}}$ depend from the target density.

5) Annihilations at NTP in coincidence with X-rays of protonium, which is characterised by the fractions of annihilation ${}^{NTP}f_{2J^{PC}}$, correspond to a specific distribution of J^{PC} P-wave initial states, that is in general different from the distribution of P-wave initial states at NTP, which in turn is characterised by the annihilation fractions ${}^{NTP}F_{J^{PC}}$.

6) The assumption made in the ASTERIX analysis that annihilation with L X-rays of protonium in coincidence provides a sample of "pure" P-wave annihilation is arbitrary. There is no ideal target where one can have "pure" P-wave or "pure" S-wave annihilation. In the conditions of the ASTERIX experiment the request of X-rays in coincidence was surely providing a sample of events were annihilations occurred dominantly from the four P-wave sublevels of the n=2 state of protonium. However the ratios between the fractions of annihilations of n=2, 1^{+-}, 0^{++}, 1^{++} and 2^{++} levels in a target at NTP are not known and need not be proportional to the annihilation fractions at NTP for $J^{PC} = 1^{+-}$, 0^{++}. 1^{++} and 2^{++}.

5. CONCLUSIONS

We have shown that the annihilation fractions and the branching ratios for annihilations into exclusive channels can be determined by measurements of fractions of annihilations at

several target densities with a method that is independent from hypothesis about the effects of strong interactions on the levels of protonium.

Six target pressures are required to fulfil this program without making any assumptions about the dynamics of the atomic cascade of protonium.

Under the hypothesis that the collisional effects, that make the cascade density dependent, do not induce transitions between spin triplet and spin singlet states, two target densities are necessary and sufficient to measure the annihilation fractions of the two types of singlet states and branching ratios of annihilations from singlet states. Under the same hypothesis four target densities are necessary and sufficient to measure the four annihilation fractions of the triplet states and branching ratios of annihilations from triplet states. Measurements at four densities of more than six branching ratios will provide checks of the validity of the above mentioned hypothesis of conservation of the total spin of protonium until the end of the atomic cascade.

We have also shown that assumptions made implicitly in the available derivations of the relative fractions of S- and P-wave annihilation are not justified and may have caused errors, which cannot be estimated at present specially because the total annihilation widths of the four $n=2$ P-wave levels are not known.

We have used the term branching ratio to indicate quantities that are not dependent from the target density, and we have used the term fraction for quantities that do depend from the target density.

We have defined branching ratios so that the sum of the branching ratios of all annihilation channels open to a state of quantum numbers J^{PC} adds up to one. Also the sum of all annihilation fractions adds up to one. The fraction of annihilations into a channel determined experimentally using a specific target is called channel branching ratio in some papers. This way of calling is confusing, because the quantity indicated is density dependent. It is important that the target conditions (and the conditions of selection of initial states, if there are any) in association to any measurement of a fraction of annihilation channel are spelled out explicitly in order to make the experimental data usable to deduce branching ratios and annihilation fractions.

A number of measurements of the fractions of annihilations for exclusive channels with charged particles in the final state are available from bubble chamber measurements for a liquid H_2 target. ASTERIX has measured a set of branching fractions in H_2 gas at NTP for exclusive final states containing charged particles. Currently three experiments are taking data at LEAR [45] on $p\bar{p}$ annihilations at rest using H_2 targets of three different densities: Crystal Barrel uses a liquid H_2 target and has in the analysis pipeline several branching fractions of exclusive channels with neutrals in the final state; CP LEAR uses a gas H_2 target at normal temperature and 15 atm. pressure; OBELIX uses a H_2 gas target at NTP, and can exploit efficiently also a target of lower density. The three experiments are all equipped with detection systems capable of measuring fractions of annihilations of fully reconstructed final states. The detectors have different specialisation, but a number of final states can be measured by each of them at the same target density for cross check of systematic effects. The conditions exist therefore in order that the necessary good statistics data be collected, so that the annihilation fractions can be determined at the densities for which they are more urgently needed. Angular distributions of the detected particles are needed in order to measure the J^{PC} fractions of annihilations.

7. Acknowledgements

REFERENCES

[1] ASTERIX Collaboration, M. Doser et al., Nucl. Phys. **A486** (1988) 493.

[2] ASTERIX Collaboration, M. Doser et al., Phys. Lett. **B215** (1988) 792.

[3] For bubble chamber data see the review: R. Armenteros and B. French, Antinucleon–nucleon interactions, *in* High Energy Physics, ed. E.H.S. Burhop (Academic Press, New York, 1969), Vol. 4, pp. 237–417.

[4] CP LEAR Collaboration, R. Adler et al., Phys. Lett. **B267** (1991) 154;
OBELIX Collaboration, A. Adamo et al., Phys. Lett. **B284** (1992) 448.

[5] E. Auld et al., Phys. Lett. **77B** (1978) 454.

[6] ASTERIX Collaboration, S. Ahmad et al., Phys. Lett. **157B** (1985) 333.
ASTERIX Collaboration, M. Ziegler et al., Phys. Lett. **206B** (1988) 151.

[7] C.A. Baker et al., Nucl. Phys. **A483** (1988) 631.

[8] C.W.E. von Ejik et al., Nucl. Phys. **A486** (1988) 604.

[9] R. Bacher et al., Z. Phys. **A334** (1989) 93.

[10] ASTERIX Collaboration, U. Schaefer et al., Nucl. Phys. **A495** (1989) 451.

[11] R. Bacher et al., 'High precision measurement of antiprotonic hydrogen and deuterium X-rays', *in* Low Energy Antiproton Physics (LEAP '90 Proceedings) (eds. P. Carlson, A. Kerk and S. Szilagyi) World Scientific 1991, p. 373.
See also K. Heitlinger et al., CERN–PPE/91-161 preprint.

[12] U. Gastaldi, 'q$\bar{q}$ and exotic mesons from annihilations of the p$\bar{p}$ atom at LEAR,' *in* Problems of fundamental modern physics II, eds. R. Cherubini, P. Dalpiaz and B. Minetti (World Scientific, Singapore, 1991) pp. 59–84.

[13] S. Devons et al., Phys. Rev. Lett. **27** (1971) 1614.

[14] G. Bassompierre et al., in: Proc. 4th European Antiproton Symposium, Barr, 1978, ed. A. Fridman (Editions du CNRS, Paris, 1979), p. 139.

[15] L. Adiels et al., Z. Phys. **C35** (1987) 15.

[16] M. Chiba et al., Phys. Lett. **202B** (1988) 447 and Phys. Rev. **D38** (1989) 2021.

[17] Crystal Barrel Collaboration, 'p$\bar{p}$ annihilation: two-body branching ratios from Crystal Barrel', presented by A. Noble at the LEAP '92 Conference, Courmayeur, September 1992.

[18] ASTERIX Collaboration, B. May et al., Phys. Lett. **B225** (1989) 450.

[19] ASTERIX Collaboration, K.D. Duch et al., Z. Phys. **C45** (1989) 223.

[20] ASTERIX Collaboration, B. May et al., Z. Phys. **C46** (1990) 191.

[21] ASTERIX Collaboration, B. May et al., Z. Phys. **C46** (1990) 203.

[22] ASTERIX Collaboration, P. Weidenhauer et al., Z. Phys. **C47** (1990) 353.

[23] ASTERIX Collaboration, J. Reifenrother et al., Phys. Lett. **B267** (1991) 299.

[24] G. Bassompierre et al., Phys. Lett. **64B** (1976) 475 and **68B** (1977) 477.

[25] G. Bardin et al., Phys. Lett. **B255** (1991) 149.

[26] U. Gastaldi, Nucl. Phys. **23B** (1991) 224–262, see in particular p. 227, 233, 238, 248 and 252.

[27] U. Gastaldi et al., 'Spin polarization in p$\bar{p}$ annihilations at rest', LNL–OXSOFF Note UG 03/91, presented at the NAN '91 Workshop, Moscow 8–11 July 1991.

[28] U. Gastaldi et al., 'Physics from studies of p$\bar{p}$ annihilations in coincidence with protonium X-rays', in preparation.

[29] A. Zichichi, S.M. Berman, N. Cabibbo and R. Gatto, Nuovo Cimento **24** (1962) 170.

[30] U. Gastaldi, Lecture Notes in Physics **273** (1987) 503.

[31] U. Gastaldi, in Spectrocopy of Light and Heavy Quarks, Proc. Int. School of Physics with Low Energy Antiprotons, Erice, 1987 (eds. U. Gastaldi, R. Klapisch and F. Close) (Plenum Press, New York, 1989) p. 311–339.

[32] C. Amsler and F. Myhrer, Ann. Rev. Nucl. Part. Science **41** (1991) 219.

[33] E. Klempt, in same Proc. as Ref. 11, p. 273.

[34] C. Dover, T. Gutsche, M. Maruyama and A. Faessler, 'The physics of nucleon–antinucleon annihilation', BNL-47212 preprint, 1992, to appear in Prog. Part. Nucl. Physics.

[35] T.B. Day, G.A. Snow and J. Sucher, Phys. Rev. Lett. **3** (1959) and Phys. Rev. **118** (1960) 864.

[36] B.R. Desai, Phys. Rev. **119** (1960) 1385.

[37] M. Leon and H.A. Bethe, Phys. Rev. **127** (1962) 636.

[38] U. Gastaldi, 'Low and High resolution protonium spectroscopy at LEAR in ACOL time', *in* Fundamental Symmetries, eds. Ph. Bloch et al. (Plenum Press, New York, 1987). pp. 307–327.

[39] C. Batty, Rep. Prog. Phys. **52** (1989) 1165.

[40] C. Batty, Nucl. Phys. **A508** (1990) 894.

[41] G. Reifenrother and E. Klempt, Nuclear Phys. **A503** (1989) 885.

[42] J.M. Richard and M.E. Sainio, Phys. Lett. **110B** (1982) 349.

[43] See Ref. 3 and Refs. 28–23 for channels with charged mesons in the final state.

For channels with only γ's in the final state see Crystal Barrel experiment, Iad. Phys. **55** (1992) 1396, presented by U. Wiedner at the NAN '91 Workshop, Moscow, ITEP, 8-11 July 1991, and Refs. therein.

[44] R. Bizzarri, Nuovo Cimento **53A** (1968) 956.

[45] Experiments at CERN 1991.

Table 1

Decomposition of the fractions of annihilation f^{ch} of a number of channels into their JPC components $f^{ch}_{J^{PC}}$ for one fixed target density ρ. The symbol ρ is not put on the top left of the F and f terms for briefness.

$$B^{3\pi^0}_{0^{-+}}F_{0^{-+}} + B^{3\pi^0}_{1^{++}}F_{1^{++}} + B^{3\pi^0}_{2^{++}}F_{2^{++}} = f^{3\pi^0}_{0^{-+}} + f^{3\pi^0}_{1^{++}} + f^{3\pi^0}_{2^{++}} = f^{3\pi^0}$$

$$B^{2\pi^0\eta}_{0^{-+}}F_{0^{-+}} + B^{2\pi^0\eta}_{1^{++}}F_{1^{++}} + B^{2\pi^0\eta}_{2^{++}}F_{2^{++}} = f^{2\pi^0\eta}_{0^{-+}} + f^{2\pi^0\eta}_{1^{++}} + f^{2\pi^0\eta}_{2^{++}} = f^{2\pi^0\eta}$$

$$B^{\pi^0 2\eta}_{0^{-+}}F_{0^{-+}} + B^{\pi^0 2\eta}_{1^{++}}F_{1^{++}} + B^{\pi^0 2\eta}_{2^{++}}F_{2^{++}} = f^{\pi^0 2\eta}_{0^{-+}} + f^{\pi^0 2\eta}_{1^{++}} + f^{\pi^0 2\eta}_{2^{++}} = f^{\pi^0 2\eta}$$

$$B^{\omega\omega}_{0^{-+}}F_{0^{-+}} + B^{\omega\omega}_{0^{++}}F_{0^{++}} + B^{\omega\omega}_{1^{++}}F_{1^{++}} + B^{\omega\omega}_{2^{++}}F_{2^{++}} = f^{\omega\omega}_{0^{-+}} + f^{\omega\omega}_{0^{++}} + f^{\omega\omega}_{1^{++}} + f^{\omega\omega}_{2^{++}} = f^{\omega\omega}$$

$$B^{\omega\phi}_{0^{-+}}F_{0^{-+}} + B^{\omega\phi}_{0^{++}}F_{0^{++}} + B^{\omega\phi}_{1^{++}}F_{1^{++}} + B^{\omega\phi}_{2^{++}}F_{2^{++}} = f^{\omega\phi}_{0^{-+}} + f^{\omega\phi}_{0^{++}} + f^{\omega\phi}_{1^{++}} + f^{\omega\phi}_{2^{++}} = f^{\omega\phi}$$

$$B^{K_S K_L}_{1^{--}}F_{1^{--}} = f^{K_S K_L}_{1^{--}} = f^{K_S K_L}$$

$$B^{\omega 2\pi^0}_{1^{--}}F_{1^{--}} + B^{\omega 2\pi^0}_{1^{+-}}F_{1^{+-}} = f^{\omega 2\pi^0}_{1^{--}} + f^{\omega 2\pi^0}_{1^{+-}} = f^{\omega 2\pi^0}$$

$$B^{\pi^0\omega}_{1^{--}}F_{1^{--}} + B^{\pi^0\omega}_{1^{+-}}F_{1^{+-}} = f^{\pi^0\omega}_{1^{--}} + f^{\pi^0\omega}_{1^{+-}} = f^{\pi^0\omega}$$

$$B^{\pi^0\phi}_{1^{--}}F_{1^{--}} + B^{\pi^0\phi}_{1^{+-}}F_{1^{+-}} = f^{\pi^0\phi}_{1^{--}} + f^{\pi^0\phi}_{1^{+-}} = f^{\pi^0\phi}$$

$$B^{\eta\omega}_{1^{--}}F_{1^{--}} + B^{\eta\omega}_{1^{+-}}F_{1^{+-}} = f^{\eta\omega}_{1^{--}} + f^{\eta\omega}_{1^{+-}} = f^{\eta\omega}$$

$$B^{\eta\phi}_{1^{--}}F_{1^{--}} + B^{\eta\phi}_{1^{+-}}F_{1^{+-}} = f^{\eta\phi}_{1^{--}} + f^{\eta\phi}_{1^{+-}} = f^{\eta\phi}$$

$$B^{\eta'\omega}_{1^{--}}F_{1^{--}} + B^{\eta'\omega}_{1^{+-}}F_{1^{+-}} = f^{\eta'\omega}_{1^{--}} + f^{\eta'\omega}_{1^{+-}} = f^{\eta'\omega}$$

$$B^{K^+K^-}_{1^{--}}F_{1^{--}} + B^{K^+K^-}_{0^{++}}F_{0^{++}} + B^{K^+K^-}_{2^{++}}F_{2^{++}} = f^{K^+K^-}_{1^{--}} + f^{K^+K^-}_{0^{++}} + f^{K^+K^-}_{2^{++}} = f^{K^+K^-}$$

$$B^{\pi^+\pi^-}_{1^{--}}F_{1^{--}} + B^{\pi^+\pi^-}_{0^{++}}F_{0^{++}} + B^{\pi^+\pi^-}_{2^{++}}F_{2^{++}} = f^{\pi^+\pi^-}_{1^{--}} + f^{\pi^+\pi^-}_{0^{++}} + f^{\pi^+\pi^-}_{2^{++}} = f^{\pi^+\pi^-}$$

$$B^{\pi^0\pi^0}_{0^{++}}F_{0^{++}} + B^{\pi^0\pi^0}_{2^{++}}F_{2^{++}} = f^{\pi^0\pi^0}_{0^{++}} + f^{\pi^0\pi^0}_{2^{++}} = f^{\pi^0\pi^0}$$

$$B^{\pi^0\eta}_{0^{++}}F_{0^{++}} + B^{\pi^0\eta}_{2^{++}}F_{2^{++}} = f^{\pi^0\eta}_{0^{++}} + f^{\pi^0\eta}_{2^{++}} = f^{\pi^0\eta}$$

$$B^{\pi^0\eta'}_{0^{++}}F_{0^{++}} + B^{\pi^0\eta'}_{2^{++}}F_{2^{++}} = f^{\pi^0\eta'}_{0^{++}} + f^{\pi^0\eta'}_{2^{++}} = f^{\pi^0\eta'}$$

$$B^{\eta\eta}_{0^{++}}F_{0^{++}} + B^{\eta\eta}_{2^{++}}F_{2^{++}} = f^{\eta\eta}_{0^{++}} + f^{\eta\eta}_{2^{++}} = f^{\eta\eta}$$

$$B^{\eta\eta'}_{0^{++}}F_{0^{++}} + B^{\eta\eta'}_{2^{++}}F_{2^{++}} = f^{\eta\eta'}_{0^{++}} + f^{\eta\eta'}_{2^{++}} = f^{\eta\eta'}$$

$$B^{K_S K_S}_{0^{++}}F_{0^{++}} + B^{K_S K_S}_{2^{++}}F_{2^{++}} = f^{K_S K_S}_{0^{++}} + f^{K_S K_S}_{2^{++}} = f^{K_S K_S}$$