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LAGRANGIAN OF AN ELASTIC SPHERE IN INTERACTION WITH N HARMONIC OSCILLATORS

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1. Introduction

Gravitational wave detection is a research field which is actively under development by many groups all over the world. Beside the cylindrical bar detectors (two of which, CERN (Geneva) and Baton Rouge (Louisiana), are in operation and other two, Rome and Legnaro, are under construction) also spherically shaped bodies can be used as detectors of gravitational waves. The physics of spherical antennae is not yet completely developed, specially for what concerns the response of the electro-mechanical detectors used for transducing mechanical into electric signals.

Purpose of this internal report is to solve exactly the mechanical problem of an elastic sphere with N attached transducers. We in fact are able to determine the spectrum of eigenfrequencies and eigenfunctions of the full apparatus. Our results are cast in a form which is suitable for subsequent numerical simulations. The importance of the latter is due to the fact that one can compare cylindrical versus spherical antennae and get a precise idea about their performances as gravitational wave detectors.

In section 2 the equations of motion for an elastic free sphere are written and the spectrum of its eigenfrequencies are determined. In section 3 the lagrangian for the full problem is set up. In section 4 the coupling of the sphere with N harmonic oscillators (transducers) is studied. In section 5 the resonance mode $l = 2$ (i.e. the mode of the sphere which couples directly with a gravitational wave) is studied in details and future work is briefly sketched.

2. Equations of motion of a free elastic sphere

If we call $\boldsymbol{u}(\boldsymbol{x})$ the displacement from the equilibrium position at point $\boldsymbol{x}$ one has (see (3.11) of [1])

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu\ \overline{\nabla}^2 \boldsymbol{u} + (\lambda + \mu)\ \overline{\nabla}(\overline{\nabla} \cdot \boldsymbol{u}) \tag{2.1}$$

Using the well known decomposition

$$u = u^{(t)} + u^{(l)}$$

where $u^{(t)}$ is a divergence free field and $u^{(l)}$ a rotation free one, i.e.

$$\overline{\nabla} \cdot u^{(t)} = 0 \qquad \overline{\nabla} \times u^{(l)} = 0 \tag{2.2}$$

eq.(2.1) is splitted into two wave equations, one for the transverse field $u^{(t)}$, characterized by transverse propagation velocity $c_t = (\mu/\rho)^{1/2}$ and another one for the longitudinal field $u^{(l)}$, characterized by longitudinal propagation velocity $c_l = [(\lambda + 2\mu)/\rho]^{1/2}$. We notice that the words longitudinal and transversal do not refer to the geometry of the oscillations but only to the properties of being divergence and rotation free. Finally we can say that the equations of motion of an elastic body are the following two

$$\Delta u^{(t)} - \frac{1}{c_t^2}\frac{\partial^2}{\partial t^2} u^{(t)} = 0 \qquad c_t = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \tag{2.3}$$

$$\Delta u^{(l)} - \frac{1}{c_l^2}\frac{\partial^2}{\partial t^2} u^{(l)} = 0 \qquad c_l = \left[\frac{(\lambda + 2\mu)}{\rho}\right]^{\frac{1}{2}} \tag{2.4}$$

Because of eq. (2.2) the fields $u^{(t)}$ and $u^{(l)}$ can be derived from a vector potential χ and a scalar potential ϕ respectively by means of

$$u^{(t)} = \overline{\nabla} \times \chi \tag{2.5}$$

$$u^{(l)} = \overline{\nabla}\phi \tag{2.6}$$

Eqs. (2.4), (2.6) show that the potential ϕ satisfies the scalar wave equation with velocity c_l

$$\Delta\phi - \frac{1}{c_l^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{2.7}$$

A solution which is regular at the origin can be written as

$$\phi(x, t) = \sum_{nlm} \frac{1}{q_{nl}} A^{(s)}_{nlm}(t)\; j_l(q_{nl} r)\; Y_{lm}(\theta, \phi) \tag{2.8}$$

where, putting

$$q_{nl}\ c_l = \omega_{nl} \tag{2.9}$$

the coefficients $A^{(s)}_{nlm}$ satisfy the equations

$$\ddot{A}^{(s)}_{nlm} + \omega^2_{nl}\ A^{(s)}_{nlm} = 0 \tag{2.10}$$

where q_{nl} depends on boundary conditions. From this one sees that the longitudinal part of the displacement can be written as

$$\boldsymbol{u}^{(l)}(\boldsymbol{x}, t) = \sum_{nlm} \frac{1}{q_{nl}}\ A^{(s)}_{nlm}(t)\ \overline{\nabla}\,[j_l(q_{nl}r)\ Y_{lm}(\theta, \phi)] \tag{2.11}$$

Let us now consider the transverse displacement. From the general vector formula

$$\Delta\,(\boldsymbol{r}\cdot\boldsymbol{A}) = \boldsymbol{r}\cdot(\Delta\boldsymbol{A}) + 2\ \overline{\nabla}\cdot\boldsymbol{A} \tag{2.12}$$

one gets for $\boldsymbol{u}^{(t)}$ (see eq.(2.2))

$$\Delta\left(\boldsymbol{r}\cdot\boldsymbol{u}^{(t)}\right) = \boldsymbol{r}\cdot\left(\Delta\boldsymbol{u}^{(t)}\right) \tag{2.13}$$

i.e. the scalar $\boldsymbol{r}\cdot\boldsymbol{u}^{(t)}$ satisfies the scalar wave equation with velocity c_t. We can write the general solution as

$$\boldsymbol{r}\cdot\boldsymbol{u}^{(t)} = \sum_{nlm} \frac{l(l+1)}{k_{nl}}\ A^{(t)}_{nlm}(t)\ j_l(k_{nl}r)\ Y_{lm}(\theta, \phi) \tag{2.14}$$

Now eq. (2.5) implies

$$\boldsymbol{r}\cdot\boldsymbol{u}^{(t)} = \boldsymbol{r}\cdot(\overline{\nabla}\times\chi) = (\boldsymbol{r}\times\overline{\nabla})\cdot\chi = i\boldsymbol{L}\cdot\chi \tag{2.15}$$

from which

$$\boldsymbol{L}\cdot\chi = \sum_{nlm} \frac{l(l+1)}{ik_{nl}}\ A^{(t)}_{nlm}(t)\ j_l(k_{nl}r)\ Y_{lm}(\theta, \phi) \tag{2.16}$$

where $\boldsymbol{L}$ acts only on the angular variables and obeys the following equation

$$\boldsymbol{L}^2\ Y_{lm} = l(l+1)\ Y_{lm} \tag{2.17}$$

Therefore

$$\chi = \sum_{nlm} \frac{1}{ik_{nl}} \; A^{(t)}_{nlm}(t) \; j_l(k_{nl}r) \; \boldsymbol{L}Y_{lm}(\theta,\phi) \tag{2.18}$$

from which

$$\boldsymbol{u}^{(t)} = \sum_{nlm} \frac{1}{ik_{nl}} \; A^{(t)}_{nlm}(t) \; \overline{\nabla} \times [j_l(k_{nl}r) \; \boldsymbol{L}Y_{lm}(\theta,\phi)] \tag{2.19}$$

where the coefficients of the development $A^{(t)}_{nlm}(t)$ satisfy the equation

$$\ddot{A}^{(t)}_{nlm} + \omega^2_{nl} \; A^{(t)}_{nlm} = 0 \tag{2.20}$$

$$\omega_{nl} = k_{nl} \; c_t \tag{2.21}$$

Finally we have

$$\begin{aligned} \boldsymbol{u} = \boldsymbol{u}^{(l)} + \boldsymbol{u}^{(t)} = \sum_{nlm} \Bigg\{ & \frac{1}{q_{nl}} \; A^{(s)}_{nlm}(t) \; \overline{\nabla} [j_l(q_{nl}r) \; Y_{lm}(\theta,\phi)] + \\ & + \frac{1}{ik_{nl}} \; A^{(t)}_{nlm}(t) \; \overline{\nabla} \times [j_l(k_{nl}r) \; \boldsymbol{L}Y_{lm}(\theta,\phi)] \Bigg\} \end{aligned} \tag{2.22}$$

We want now to write $\boldsymbol{u}(\boldsymbol{x},t)$ as a sum of a vector term parallel to $\boldsymbol{n} = \boldsymbol{r}/|\boldsymbol{r}|$ and another parallel to $\overline{\nabla} Y_{lm}$. One gets

$$\overline{\nabla} [j_l(q_{nl}r) \; Y_{lm}(\theta,\phi)] = \left[\frac{\partial j_l(q_{nl}r)}{\partial r} \; \boldsymbol{n} + j_l(q_{nl}r) \overline{\nabla} \right] \; Y_{lm}(\theta,\phi) \tag{2.23}$$

and

$$\begin{aligned} & \overline{\nabla} \times [j_l(k_{nl}r) \; \boldsymbol{L}Y_{lm}(\theta,\phi)] = \\ & \quad = \overline{\nabla} j_l(k_{nl}r) \times \boldsymbol{L}Y_{lm}(\theta,\phi) + j_l(k_{nl}r) \overline{\nabla} \times \boldsymbol{L}Y_{lm}(\theta,\phi) = \\ & \quad = \frac{\partial j_l(q_{nl}r)}{\partial r} \; \boldsymbol{n} \times \boldsymbol{L} \; Y_{lm}(\theta,\phi) + j_l(k_{nl}r) \overline{\nabla} \times \boldsymbol{L}Y_{lm}(\theta,\phi) \end{aligned} \tag{2.24}$$

For the first term of the preceding equation we have

$$\begin{aligned} \boldsymbol{n} \times \boldsymbol{L} \; Y_{lm}(\theta,\phi) = & \frac{1}{i} \boldsymbol{n} \times (\boldsymbol{r} \times \overline{\nabla} Y_{lm}) = \\ & \frac{1}{i} \left[(\boldsymbol{n} \cdot \overline{\nabla} Y_{lm}) \; \boldsymbol{r} - (\boldsymbol{n} \cdot \boldsymbol{r}) \; \overline{\nabla} Y_{lm} \right] \end{aligned}$$

from which it follows

$$\boldsymbol{n} \times \boldsymbol{L}\, Y_{lm}(\theta,\phi) = ir\overline{\nabla}Y_{lm} \tag{2.25}$$

For the second term of eq. (2.24) we have

$$\overline{\nabla} \times \boldsymbol{L} Y_{lm}(\theta,\phi) = \frac{1}{i}\overline{\nabla} \times \left[\boldsymbol{r} \times \overline{\nabla} Y_{lm}(\theta,\phi)\right] =$$

$$\frac{1}{i}\left\{\boldsymbol{r}\left(\overline{\nabla}\cdot\overline{\nabla}Y_{lm}\right) - \overline{\nabla}Y_{lm}\left(\overline{\nabla}\cdot\boldsymbol{r}\right) + \left(\overline{\nabla}Y_{lm}\cdot\overline{\nabla}\right)\boldsymbol{r} - \left(\boldsymbol{r}\cdot\overline{\nabla}\right)\overline{\nabla}Y_{lm}\right\}$$

Let us consider the four terms separately.

$$\boldsymbol{r}\ \left(\overline{\nabla}\cdot\overline{\nabla}Y_{lm}\right) = \boldsymbol{r}\ \overline{\nabla}^2 Y_{lm} = r\boldsymbol{n}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\ Y_{lm}(\theta,\phi)\right) - \frac{L^2}{r^2}\ Y_{lm}(\theta,\phi)\right]$$

i.e.

$$\boldsymbol{r}\ \left(\overline{\nabla}\cdot\overline{\nabla}Y_{lm}\right) = -\frac{l(l+1)}{r}\ \boldsymbol{n}\ Y_{lm}$$

and for the other three

$$-\overline{\nabla}\ Y_{lm}\left(\overline{\nabla}\cdot\boldsymbol{r}\right) = -3\ \overline{\nabla}Y_{lm}$$

$$\left(\overline{\nabla}\ Y_{lm}\cdot\overline{\nabla}\right)\ \boldsymbol{r} = \left(\overline{\nabla}\ Y_{lm}\cdot\overline{\nabla}\right)\ (r\boldsymbol{n}) = \left(v_2\frac{\partial}{\partial\theta} + v_3\frac{\partial}{\partial\rho}\right)(r\boldsymbol{n}) =$$

$$r\ \left(\overline{\nabla}\ Y_{lm}\cdot\overline{\nabla}\right)\boldsymbol{n} = \overline{\nabla}\ Y_{lm}$$

$$-\left(\boldsymbol{r}\cdot\overline{\nabla}\right)\overline{\nabla}Y_{lm} = -r\frac{\partial}{\partial r}\overline{\nabla}\ Y_{lm} = \overline{\nabla}\ Y_{lm}$$

from which

$$\overline{\nabla}\times\boldsymbol{L}\ Y_{lm}(\theta,\phi) = i\ \left[\frac{l(l+1)}{r}\boldsymbol{n}\ + \overline{\nabla}\right]\ Y_{lm}(\theta,\phi) \tag{2.26}$$

Finally we get

$$\overline{\nabla}\times\left[j_l(k_{nl}r)\ \boldsymbol{L}\ Y_{lm}(\theta,\phi)\right] =$$
$$i\left\{\frac{l(l+1)}{r}\ j_l(k_{nl}r)\ \boldsymbol{n} + \frac{\partial}{\partial r}\left[rj_l(k_{nl}r)\right]\overline{\nabla}\right\}\ Y_{lm}(\theta,\phi) \tag{2.27}$$

Therefore

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{nlm}\left\{\frac{A^{(s)}_{nlm}}{q_{nl}}\left[\frac{\partial j_l(q_{nl}r)}{\partial r}\boldsymbol{n} + j_l(q_{nl}r)\overline{\nabla}\right] + \right.$$
$$\left.\frac{A^{(t)}_{nlm}}{k_{nl}}\left[\frac{l(l+1)}{r}j_l(k_{nl}r)\,\boldsymbol{n} + \frac{\partial}{\partial r}\left(rj_l(k_{nl}r)\right)\overline{\nabla}\right]\right\}Y_{lm}(\theta,\phi) \tag{2.28}$$

Setting

$$A^{(s)}_{nlm}(t) = A_{nlm}(t)\, s_{nlm}$$
$$A^{(t)}_{nlm}(t) = A_{nlm}(t)\, t_{nlm} \tag{2.29}$$

one gets

$$u(\boldsymbol{x},t) = \sum_{nlm} A_{nlm}(t) \left[a_{nlm}(r) + b_{nlm}(r)\, R\overline{\nabla}\right] Y_{lm}(\theta,\phi) \tag{2.30}$$

where

$$a_{nlm}(r) = \frac{s_{nlm}}{q_{nl}} \frac{\partial j_l(q_{nl}r)}{\partial r} + \frac{t_{nlm}}{k_{nl}} \frac{l(l+1)}{r} j_l(k_{nl}r)$$
$$b_{nlm}(r) = \left[\frac{s_{nlm}}{q_{nl}} j_l(q_{nl}r) + \frac{t_{nlm}}{k_{nl}} \frac{\partial}{\partial r}(r j_l(k_{nl}r))\right] \frac{1}{R} \tag{2.31}$$

As boundary conditions are applied for $r = R$ it is convenient to set

$$q_{nl} = \frac{x_{nl}}{R}$$

and therefore also (see eqs. (2.1) and (2.24))

$$k_{nl} = \frac{c_l}{c_t} \frac{x_{nl}}{R}$$

and also

$$c_{nlm} = \frac{s_{nlm}}{x_{nL}} \qquad d_{nlm} = \frac{t_{nlm}}{x_{nL}} \frac{c_t}{c_l} \tag{2.32}$$

$$a_{nlm}(r) = c_{nlm}\, R\, \frac{dj_l(q_{nl}r)}{dr} + d_{nlm}\, R\, \frac{l(l+1)}{r} j_l(k_{nl}r)$$
$$b_{nlm}(r) = c_{nlm} j_l(q_{nl}r) + d_{nlm} \frac{d}{dr}(r j_l(k_{nl}r)) \tag{2.33}$$

In order to get the eigenfrequencies of the sphere we must apply suitable boundary conditions. For a free body these are equivalent to assume that the total force per unit area at the surface of the elastic medium is zero in the normal direction. If n_γ are the components of the unit normal vector this means the following three conditions

$$n_\gamma \sigma_{\gamma\mu} = 0 \tag{2.34}$$

where $\sigma_{\mu\nu}$ is linked to the deformation $u_{\mu\nu} = (1/2)(u_{\mu,\nu} + u_{\nu,\mu})$ by means of

$$\sigma_{\mu\nu} = \delta_{\mu\nu} \; \lambda \; u_{\gamma\gamma} + 2 \; \mu \; u_{\mu\nu}; \tag{2.35}$$

in spherical coordinates

$$\begin{aligned} \sigma_{rr} &= 2 \; \mu \; u_{rr} + \lambda \; \overline{\nabla} \cdot \boldsymbol{u} \\ \sigma_{r\theta} &= 2 \; \mu \; u_{r\theta} \\ \sigma_{r\phi} &= 2 \; \mu \; u_{r\phi} \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} u_{rr} &= \frac{\partial u_r}{\partial r} \\ u_{r\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) \\ u_{r\phi} &= \frac{1}{2}\left(\frac{1}{r sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) \end{aligned} \tag{2.37}$$

Equations (2.35) are written on the surface of the sphere $r = R$ as (see ref. [1])

$$\begin{aligned} 2 \; \mu \; u_{rr} + \lambda \; \overline{\nabla} \cdot \boldsymbol{u} &= 0 \\ u_{r\theta} &= 0 \\ u_{r\phi} &= 0 \end{aligned} \tag{2.38}$$

In order to get the eigenfrequency equations it is necessary to give $\boldsymbol{u}$ explicitly. As we have seen in (2.30), (2.33) one has:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{nlm} A_{nlm}(t) \left[a_{nlm}(r)\boldsymbol{n} + b_{nlm}(r) R \overline{\nabla}\right] Y_{lm}(\theta, \phi) \tag{2.39}$$

where

$$\begin{aligned} a_{nlm}(r) &= c_{nlm} R \frac{d}{dr} j_l(q_{nl} r) + d_{nlm} R \frac{l(l+1)}{r} j_l(k_{nl} r) \\ b_{nlm}(r) &= c_{nlm} j_l(q_{nl} r) + d_{nlm} \frac{d}{dr}\left[r j_l(k_{nl} r)\right] \end{aligned} \tag{2.40}$$

To simplify our notations we omit indices and remember that the equations are valid for any value of (n, l, m). Starting from the first of (2.38) we get:

$$
\begin{aligned}
u_{rr} &= A\frac{d}{dr}aY \\
\overline{\nabla}\cdot\boldsymbol{u} &= A\overline{\nabla}\left[aY\boldsymbol{n} + Rb\overline{\nabla}Y\right] = \\
&= A\boldsymbol{n}\cdot\overline{\nabla}(aY) + AaY\overline{\nabla}\cdot\boldsymbol{n} + AR\overline{\nabla}Y\cdot\overline{\nabla}b + Rb\overline{\nabla}^2Y = \\
&= A(\boldsymbol{n}\cdot\overline{\nabla}a)Y + Aa(\boldsymbol{n}\cdot\overline{\nabla}Y) + A\,\frac{2}{r}\,aY - AbR\frac{l(l+1)}{r^2}Y = \\
&= A\left[\frac{\partial u}{\partial r} + 2\frac{a}{r} - Rb\frac{l(l+1)}{r^2}\right]Y
\end{aligned}
\tag{2.41}
$$

The third equality in the second equation is due to the orthogonality condition $\overline{\nabla}Y\cdot\overline{\nabla}b = 0$, $\overline{\nabla}^2Y = -l(l+1)Y/r^2$ and $\overline{\nabla}\cdot\boldsymbol{n} = 2/r$, while the fourth is due to $\boldsymbol{n}\cdot\overline{\nabla}a = \partial a/\partial r$ and $\boldsymbol{n}\cdot\overline{\nabla}Y = 0$. The first condition of (2.38) is therefore written as:

$$
(2\mu+\lambda)\frac{\partial a_{nlm}}{\partial r} + 2\lambda\frac{a_{nlm}}{r} - R\frac{l(l+1)}{r^2}\lambda b_{nlm} = 0 \tag{2.42}
$$

In order to make explicit the last two equations (2.38) we observe that

$$
\overline{\nabla}Y = \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial Y}{\partial\theta} + \boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial Y}{\partial\phi}
$$

Therefore:

$$
\boldsymbol{u} = A\left(aY\boldsymbol{n} + R\frac{b}{r}\frac{\partial Y}{\partial\theta}\boldsymbol{e}_\theta + \frac{Rb}{r\sin\theta}\frac{\partial Y}{\partial\phi}\boldsymbol{e}_\phi\right)
$$

Let us consider

$$
u_{r\theta} = \frac{1}{2}\left(u_{\theta,r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) = 0 \tag{2.43}
$$

Being

$$
\begin{aligned}
\frac{1}{A}u_{\theta,r} &= \left(\frac{R}{r}\frac{\partial b}{\partial r} - \frac{Rb}{r^2}\right)\frac{\partial Y}{\partial\theta} \\
-\frac{u_\theta}{r}\frac{1}{A} &= -\frac{Rb}{r^2}\frac{\partial Y}{\partial\theta} \\
\frac{1}{A}\frac{1}{r}\frac{\partial u_r}{\partial\theta} &= \frac{a}{r}\frac{\partial Y}{\partial\theta}
\end{aligned}
$$

equation (2.43) becomes

$$
\frac{R}{r}\frac{\partial b}{\partial r} - R\frac{2b}{r^2} + \frac{a}{r} = 0 \tag{2.44}
$$

Starting from

$$u_{r\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) = 0 \tag{2.45}$$

we get in a similar way

$$\frac{R}{r}\frac{\partial b}{\partial r} - R\frac{2b}{r^2} + \frac{a}{r} = 0 \tag{2.46}$$

which is the same as equation (2.44). Therefore the boundary conditions are satisfied if

$$\begin{aligned} &(a) \quad a_n + R\frac{\partial b_n}{\partial r} - 2\frac{R}{r}b_n = 0 \\ &(b) \quad (2\mu+\lambda)\frac{\partial a_n}{\partial r} + 2\lambda\frac{a_n}{r} - R\frac{l(l+1)}{r^2}\lambda b_n = 0 \end{aligned} \tag{2.47}$$

where for simplicity index n stays for all three nlm and a_n, b_n are given by (2.40). In what follows we drop all indices in the Bessel functions. Substituting (2.40) into (2.47) (a) we get

$$c_n\left\{2\frac{d}{dr}j(qr) - \frac{2}{r}j(qr)\right\} +$$
$$+d_n\left\{\frac{l(l+1)}{r}j(kr) + \frac{d^2}{r^2}[rj(kr)] - \frac{2}{r}\frac{d}{dr}[rj(kr)]\right\} = 0$$

Using the spherical Bessel function equation

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}\right]j(kr) = 0 \tag{2.48}$$

one gets

$$\frac{d^2}{dr^2}[rj(kr)] = \left[-rk^2 + \frac{l(l+1)}{r}\right]j(kr)$$

and

$$-\frac{2}{r}\frac{d}{dr}[rj(kr)] = -\frac{2}{r}j(kr) - 2\frac{d}{dr}j(kr)$$

The coefficient of d_n becomes

$$\left[\frac{l(l+1)}{r} + \frac{l(l+1)}{r} - rk^2 - \frac{2}{r} - 2\frac{d}{r}\right]j(kr) =$$
$$= 2r\left[\frac{l^2+l-1}{r^2} - \frac{k^2}{2} - \frac{1}{r}\frac{d}{dr}\right]j(kr)$$

while that of c_n can be cast in the form

$$2r\left[\frac{1}{r}\frac{d}{dr}j(qr)-\frac{1}{r^2}j(qr)\right]=2r\frac{d}{r}\left[\frac{j(qr)}{r}\right]$$

So equation (2.47) (a) becomes

$$c_n\frac{d}{dr}\left[\frac{j(qr)}{r}\right]+d_n\left[\frac{l^2+l-1}{r^2}-\frac{k^2}{2}-\frac{1}{r}\frac{d}{dr}\right]j(kr)=0 \qquad (2.49)$$

calculated for $r=R$. This equation is the same as equation (14a) of [2]. Let us now consider (2.47) (b). Substituting (2.48) we get

$$c_n\left\{(2\mu+\lambda)\frac{d^2}{dr^2}j(qr)+\frac{2\lambda}{r}\frac{d}{dr}j(qr)-\frac{\lambda}{r^2}l(l+1)j(qr)\right\}+$$
$$d_n\left\{(2\mu+\lambda)\frac{d}{dr}\left[\frac{j(kr)}{r}\right]+\frac{2\lambda}{r^2}j(kr)-\frac{\lambda}{r^2}\frac{d}{dr}\left[rj(kr)\right]\right\}l(l+1)=0$$

Using (2.48) we can rewrite the coefficient of c_n as

$$\begin{aligned}&\left\{-(2\mu+\lambda)\left[\frac{2}{r}\frac{d}{dr}+q^2-\frac{l(l+1)}{r^2}\right]+\frac{2\lambda}{r}\frac{d}{dr}-\frac{\lambda}{r^2}l(l+1)\right\}j(qr)=\\&=\left\{-2\mu\left[\frac{2}{r}\frac{d}{dr}+q^2-\frac{l(l+1)}{r^2}\right]-\lambda q^2\right\}j(qr)=\\&=2\mu\left\{\frac{l(l+1)}{r^2}-\left(1+\frac{\lambda}{2\mu}\right)q^2-\frac{2}{r}\frac{d}{dr}\right\}j(qr)\end{aligned} \qquad (2.50)$$

Now

$$\begin{aligned}\lambda&=(c_l^2-2c_t^2)\rho=\omega^2\rho\left(\frac{1}{a^2}-\frac{1}{k^2}\right)\\ \mu&=c_t^2\rho=\frac{\omega^2\rho}{k^2}\end{aligned} \qquad (2.51)$$

from which it follows

$$\left(1+\frac{\lambda}{2\mu}\right)=\frac{k^2}{2q^2} \qquad (2.52)$$

Substituting (2.52) into (2.50) one gets for the coefficient of c_n

$$2\mu\left\{\frac{l(l+1)}{r^2}-\frac{k^2}{2}-\frac{2}{r}\frac{d}{dr}\right\}j(qr)$$

Let us consider now the coefficient of d_n divided by $l(l+1)$

$$(2\mu+\lambda)\frac{d}{dr}\left[\frac{j(kr)}{r}\right]+\frac{\lambda}{r^2}\left[j(kr)-r\frac{d}{dr}j(kr)\right]=$$
$$=2\mu\left\{\left(1+\frac{\lambda}{2\mu}\right)\frac{d}{dr}+\left[\frac{j(kr)}{r}\right]+\frac{\lambda}{2\mu}\left[\frac{j(kr)}{r^2}-\frac{1}{r}\frac{d}{dr}j(kr)\right]\right\}=$$
$$=2\mu\frac{d}{dr}\left[\frac{j(kr)}{r}\right]$$

In this way (2.47) (b) becomes

$$c_n\left[\frac{l(l+1)}{r^2}-\frac{k^2}{2}-\frac{2}{r}\frac{d}{dr}\right]j(qr)+d_n l(l+1)\frac{d}{dr}\left[\frac{j(kr)}{r}\right]=0 \tag{2.53}$$

calculated for $r=R$. This equation is equal to formula (14b) of [2].

Using eqs (2.49) and (2.53) and observing that for $l=0$ eqs (2.40) give the condition

$$\left[\frac{k_{n0}^2}{2}+\frac{2}{r}\frac{d}{dr}\right]j_0(q_{n0}r)\Big|_{r=R}=0$$

(the other condition $\left[(1/r^2)+(k_{n0}^2/2)+(1/r)(d/dr)\right]j_0(k_{n0}r)=0$ gives eigenfunctions identically equal to zero). One finds

$l=0$

n	$q_{n0}R$	$\nu_{n0}\ (Hz)$
1	2.74371	2071
2	6.11676	4616

$l=1$

n	$q_{n1}R$	$\nu_{n1}\ (Hz)$
1	1.79799	1357
2	3.62074	2733
3	4.2747	3226
4	5.36478	4049

$l=2$

n	$q_{n2}R$	$\nu_{n2}\ (Hz)$
1	1.32503	1000
2	2.54929	1923
3	4.31276	3255
4	5.49285	4145
5	6.14578	4638

$l = 3$

n	$q_{n3}R$	ν_{n3} (Hz)
1	1.97509	1491
2	3.35263	2530
3	4.9906	3766

$l = 4$

n	$q_{n4}R$	ν_{n4} (Hz)
1	2.53417	1913
2	4.15425	3135
3	5.66719	4277

$l = 5$

$$q_{15} = 3.05742 \qquad \nu_{15} = 2307\ Hz$$

$l = 6$

$$q_{16} = 3.56307 \qquad \nu_{16} = 2689\ Hz$$

These results were obtained for a material with Poisson ratio $\sigma = 0.33$ for which $2c_t = c_l$. The radius of the sphere has been chosen in such a way that $\nu_{12} = 1000\ Hz$ i.e. $R \approx 1.3m$.

3. The lagrangian of the sphere

As we have seen (eq. (2.30)) the displacement from the equilibrium position of an elastic solid body in spherical coordinates can be written as (see [2])

$$\boldsymbol{u}(r, \theta, \phi, t) = \sum_{nlm} A_{nlm}(t) \boldsymbol{\psi}_{nlm}(r, \theta, \phi) \tag{3.1}$$

where

$$\boldsymbol{\psi}_{nlm}(r, \theta, \phi) = [a_{nl}(r)\boldsymbol{n} + b_{nl}(r)R\overline{\nabla}]Y_{lm}(\theta, \phi) \tag{3.2}$$

The eigenfunctions $\boldsymbol{\psi}_{nlm}(r, \theta, \phi)$ are dimensionless and orthonormal with the scalar product (* means complex conjugation)

$$< \boldsymbol{g}(r, \theta, \phi) | \boldsymbol{f}(r, \theta, \phi) > = \frac{1}{M} \int \boldsymbol{g}^* \cdot \boldsymbol{f} \rho d^3x \tag{3.3}$$

and explicitly we have

$$\frac{1}{M}\int \boldsymbol{\psi}^*_{nlm}\cdot\boldsymbol{\psi}_{n'l'm'}\rho d^3x = \delta_{nn'}\delta_{ll'}\delta_{mm'} \tag{3.4}$$

The coefficients of the developement (3.1) are therefore given by

$$<\boldsymbol{\psi}_{nlm}|\delta\boldsymbol{x}>= A_{nlm}(t) \tag{3.5}$$

Let us consider now a reference frame O'. The normal component of the displacement on the surface of a sphere in this frame is (see [2] eq. (7))

$$\boldsymbol{n}\cdot\boldsymbol{u} = \sum_{nlm'} a_{nl}(R)A'_{nlm'}(t)Y_{lm'}(\theta',\phi') \tag{3.6}$$

Now we want to express the radial displacement in another frame O rotated with respect to O' by the hour angle H and the declination δ (for the definitions of H and δ see [2]). In O we can write

$$\boldsymbol{n}\cdot\boldsymbol{u} = \sum_{nlm} a_{nl}(R)\left(\sum_{m'} A'_{nlm'}\mathcal{D}^{(l)}_{mm'}(H,\delta)\right)Y_{lm}(\theta,\phi) \tag{3.7}$$

where the operators $\mathcal{D}^{(l)}_{mm'}(H,\delta)$ are the rotation matrix for the spherical harmonics (see for instance [4]).

Let us define

$$F_{nlm} = \sum_{m'} A'_{nlm'}\mathcal{D}^{(l)}_{mm'}(H,\delta) \tag{3.8}$$

which are the natural generalizations to every order in the n and l indices of the F_m of reference [2] eq. (9). To be more precise the exact corrispondence is

$$F_m = F_{12m} \tag{3.9}$$

Because of the unitary property of the $\mathcal{D}^{(l)}_{mm'}(H,\delta)$ matrices, we can invert eq. (3.8) and find the coefficients $A'_{nlm'}$ as linear combinations of the F_{nlm}

$$A'_{nlm'} = \sum_{m} \mathcal{D}^{(l)\,*}_{mm'}(H,\delta)F_{nlm} \tag{3.10}$$

Let us now consider the kinetic energy of the sphere

$$T_S = \frac{1}{2} \int d^3x \rho \dot{u}^2 \tag{3.11}$$

Substituting eq (3.1) and taking into account eq. (3.5) one has

$$T_S = \frac{1}{2} M \sum_{nlm'} \dot{A}'_{nlm'} \dot{A}'^{*}_{nlm'} \tag{3.12}$$

Using eq. (3.10) and the unitary property for the $\mathcal{D}^{(l)}_{mm'}(H, \delta)$'s one can find T_S as a function of the generalized coordinates F_{nlm}, F^{*}_{nlm}. One gets

$$T_S = \frac{1}{2} M \sum_{nlm} \dot{F}_{nlm} \dot{F}^{*}_{nlm} \tag{3.13}$$

The equation satisfied by the coefficients $A'_{nlm'}$ is (see reference [2], eq. (5))

$$\ddot{A}'_{nlm'} + \omega^2_{nl} A'_{nlm'} = R_{nlm'}(t) \tag{3.14}$$

(we have neglected the damping) and therefore because of expression (3.12) for the kinetic energy it is easy to verify that the elastic potential energy which gives the correct motion equation should be written as

$$V_S = \frac{1}{2} M \sum_{nlm'} \omega^2_{nl} A'_{nlm'} A'^{*}_{nlm'} \tag{3.15}$$

In the same way the potential energy of the external force can be written as

$$V_{ext} = -\frac{1}{2} M \sum_{nlm'} [A'_{nlm'} R^{*}_{nlm'} + A'^{*}_{nlm'} R_{nlm'}] \tag{3.16}$$

Making use of equation (3.10) we find

$$V_S = \frac{1}{2} M \sum_{nlm'} \omega^2_{nl} F_{nlm'} F^{*}_{nlm'} \tag{3.17}$$

and

$$V_{ext} = -\frac{1}{2} M \sum_{nlm} \left[\sum_{m'} \mathcal{D}^{(l)\,*}_{mm'} F_{nlm} R^{*}_{nlm'} + \mathcal{D}^{(l)}_{mm'} F^{*}_{nlm} R_{nlm'} \right] \tag{3.18}$$

4. Coupling with N harmonic oscillators

Now we consider a set of N identical mechanical harmonic oscillators, with mass m and frequency $\omega/2\pi$, located on the surface of the sphere at the points (θ_i, ϕ_i). We call $q_i(t)$ the generalized coordinates which describe the radial motion of each oscillator with respect to the surface of the sphere.

As far as the potential energy of the oscillator is concerned, the choice of the generalized coordinates leads to the simple expression

$$V_t = \frac{m\omega^2}{2} \sum_i q_i^2 \tag{4.1}$$

As far as the kinetic energy is concerned we observe that, if we call y_i the coordinates of the i–th oscillator with respect to an inertial frame, it can be written as

$$T_t = \frac{m}{2} \sum_i \dot{y}_i^2 \tag{4.2}$$

The relation between the inertial coordinates and the generalized coordinates q_i is given by

$$q_i = y_i - \boldsymbol{n} \cdot \boldsymbol{u}(R, \theta_i, \phi_i, t)$$

Therefore, because of eq. (3.7), the following expression holds

$$y_i = q_i + \sum_{nlm} a_{nl}(R) F_{nlm} Y_{nlm}(\theta_i, \phi_i) \tag{4.3}$$

Now as a matter of convenience we define the matrix $\boldsymbol{P}^{(l)}$ with components

$$P^{(l)}_{mi} = Y^*_{lm}(\theta_i, \phi_i) \tag{4.4}$$

The connection between this matrix $\boldsymbol{P}^{(l)}$ and the $\boldsymbol{B}$ of reference [3] is given by

$$\boldsymbol{B} = \boldsymbol{P}^{(2)*} \tag{4.5}$$

Taking into account equations (4.2)–(4.4) the kinetic energy of the harmonic oscillators is

$$T_t = \frac{m}{2}\sum_i \left[\dot{q}_i^2 + \dot{q}_i \sum_{nlm} a_{nl}(R)\left(P_{mi}^{(l)\,*}\dot{F}_{nlm} + P_{mi}^{(l)}\dot{F}_{nlm}^{*}\right) + \right.$$
$$\left. \sum_{nlm}\sum_{n'l'm'} a_{nl}(R)a_{n'l'}(R)P_{mi}^{(l)\,*}P_{m'i}^{(l')}\dot{F}_{nlm}\dot{F}_{n'l'm'}^{*}\right] \tag{4.6}$$

If some external force $\tilde{f}_i(t)$ is applied on the i–th oscillator, then the potential energy of the external forces is

$$W_{ext} = -\sum_i \left[q_i + \frac{1}{2}\sum_{nlm} a_{nl}(R)\left(P_{mi}^{(l)\,*}\dot{F}_{nlm} + P_{mi}^{(l)}\dot{F}_{nlm}^{*}\right)\right]\tilde{f}_i \tag{4.7}$$

In conclusion we have found that the lagrangian of an elastic sphere subject to an external force with N mechanical oscillator is

$$L = T_S + T_t - V_S - V_{ext} - V_t - W_{ext} \tag{4.8}$$

where the various terms are defined in eqs. (3.13), (4.6), (3.17), (3.18), (4.1), and (4.7) respectively. The generalized coordinates are F_{lm}, F_{lm}^* and q_i where $l = \{1, 2, ...\}$, $m = \{-l, -l+1, ..., l\}$, and $i = \{1, 2, .., N\}$. It must be remembered that $F_{l-m} = (-1)^m F_{lm}^*$. Therefore it suffices to find the motion equations from q_i and F_{lm}^* only.

5. Resonance with l=2

In what follows we make the assumptions that the common frequency of the oscillators is equal to that of the mode $n = 1$ $l = 2$ of the sphere, that this frequency is far enough from any other one of the spectrum of the sphere and finally that the external force has only $l = 2$ components. In the case of a damping free motion the second condition is always met if all the frequencies of the spectrum of the sphere are different from the chosen one. But in real problems it is necessary also to consider frictions. If we call τ_{nlm} the damping time of the mode caracterized by the indices n, l and m then the condition is fulfilled when

$$|\omega_{12} - \omega_{nl}| >> \frac{1}{\tau} \qquad \tau = min\{\tau_{12m}, \tau_{nlm'}\}$$

In reality it must be considered that when we decouple the oscillators we find a set of eigenfrequencies that must also satisfy a relation similar to the above one as we shall see further on. For simplicity we call

$$P^{(2)}_{mi} \equiv P_{mi} \qquad F_{12m} \equiv F_m \qquad R_{12m} \equiv R_m \qquad \omega_{12} = \omega \qquad a_{12}(R) \equiv \alpha$$

Within the framework of these assumptions we find

$$T_S = \frac{1}{2} M \sum_m \dot{F}_m \dot{F}^*_m$$

$$T_t = \frac{m}{2} \sum_i \left[\dot{q}_i^2 + \dot{q}_i \sum_m \alpha \left(P^*_{mi} \dot{F}_m + P_{mi} \dot{F}^*_m \right) + \sum_{mm'} \alpha^2 P^*_{mi} P_{m'i} \dot{F}_m \dot{F}^*_{m'} \right]$$

$$V_S = \frac{1}{2} M \sum_m \omega^2 F_m F^*_m \tag{5.1}$$

$$V_{ext} = -\frac{1}{2} M \sum_m \left[\sum_{m'} \left(\mathcal{D}^{(2)*}_{mm'} F_m R^*_{m'} + \mathcal{D}^{(2)}_{mm'} F^*_m R_{m'} \right) \right]$$

$$V_t = \frac{m\omega^2}{2} \sum_i q_i^2$$

$$W_{ext} = -\sum_i \left[q_i + \frac{1}{2} \sum_m \alpha \left(P^*_{mi} \dot{F}_m + P_{mi} \dot{F}^*_m \right) \right] \tilde{f}_i$$

From eq. (5.1) and defining

$$\tilde{\boldsymbol{f}} = (\tilde{f}_i) \qquad \boldsymbol{a} = (F_m) \qquad \boldsymbol{F} = M \left(\sum_{m'} \mathcal{D}^{(2)}_{mm'} R_{2m'} \right) \tag{5.2}$$

we get the following equations of motion († means hermitian conjugation)

$$\begin{pmatrix} M\mathcal{I}_5 & 0 \\ m\alpha \boldsymbol{P}^\dagger & m\mathcal{I}_N \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{a}} \\ \ddot{\boldsymbol{q}} \end{pmatrix} + \begin{pmatrix} M\omega^2\mathcal{I}_5 & -m\omega^2\alpha \boldsymbol{P} \\ 0 & m\omega^2\mathcal{I}_N \end{pmatrix} \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} \mathcal{I}_5 & 0 \\ 0 & \mathcal{I}_N \end{pmatrix} \begin{pmatrix} \boldsymbol{F} \\ \tilde{\boldsymbol{f}} \end{pmatrix} \tag{5.3}$$

where $\mathcal{I}_n$ is the identity in n–dimensional space.

If there are no forces directly applied to the oscillators ($\tilde{\boldsymbol{f}} = 0$) but only forces f_i applied between the oscillators and the spherical surface (for instance the noise) then

$$W'_{ext} = -\sum_i q_i f_i \qquad \boldsymbol{f} = (f_i) \tag{5.4}$$

and the equations of motion become

$$\begin{pmatrix} M\mathcal{I}_5 & 0 \\ m\alpha \boldsymbol{P}^\dagger & m\mathcal{I}_N \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{a}} \\ \ddot{\boldsymbol{q}} \end{pmatrix} + \begin{pmatrix} M\omega^2\mathcal{I}_5 & -m\omega^2\alpha \boldsymbol{P} \\ 0 & m\omega^2\mathcal{I}_N \end{pmatrix} \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} \mathcal{I}_5 & -\alpha \boldsymbol{P} \\ 0 & \mathcal{I}_N \end{pmatrix} \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{f} \end{pmatrix} \tag{5.5}$$

The coefficient $\alpha = a_{12}(R)$ can be calculated using [1] and [2]. We want to stress that our eqs. (5.3) and (5.5) differ from eq. (3.1) of reference [3] because

$$\boldsymbol{P} = \boldsymbol{B}^*$$

This difference however is not essential for the general understanding of the behaviour of the system.

The next step to do is to decouple the system of $5+N$ interacting oscillators and find the eigenfrequencies. Once the calculation is performed it is very important to check if it is possible to take into account in a self–consistent way only the modes with $n = 1$ and $l = 2$. In fact if we call $\tilde{\omega}_a$ these eigenfrequencies and $\tilde{\tau}_a$ the respective damping times then the consistency relation is

$$|\tilde{\omega}_a - \omega_{nl}| >> \frac{1}{\tau} \qquad \tau = min\{\tilde{\tau}_a, \tau_{nlm}\}$$

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