

Wall Impedence for Low Moderate Beam Energies

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Abstract

Properties of the wall impedance of a storage ring are studied for low and moderate beam energies. In general, the impedance is shown to be exponentially damped after a certain frequency. The damping factor, mainly determined by a universal exponent, is derived for resistive-wall and then generalized for broad- and narrow-band structure impedances. Stability conditions for both longitudinal and transverse oscillations of a coasting cooled beam are considered.

1 Introduction

A beam travelling in a storage ring excites electromagnetic fields inside the vacuum chamber. These fields act on the beam itself and can cause beam instabilities. A linear dependence of the electromagnetic force on the beam current perturbation usually is described in terms of the wake potential or the impedance [1]. The impedance of a vacuum chamber depends on its structure, wall materials and the beam energy. The last dependence vanishes in an ultrarelativistic limit, where the solution of the Maxwell equations can be found for a beam velocity v equal to the speed of light c . This case is described in details in the monography of A. W. Chao [2]. In this limit all the fields excited by a point perturbation of the beam density, lag behind it, which is referred to as the causality principle for wake fields.

In the case of a nonrelativistic beam, its temperature is usually so high, that the coherent increments, introduced by the impedances, are much less than the Landau damping; thus, the impedance does not play any role. The situation changes for cooled beams, where at sufficiently low temperature the Landau damping is switched off. Here, to afford the stability at low temperatures the cooling rate has to be more than the maximum increment caused by the vacuum chamber impedance.

The causality principle does not work for low and moderate energies, where the relativistic factor $\gamma \simeq 1$. The reason is that in this case the Coulomb field of a particle at some impact distance r , is not a δ -function of a longitudinal coordinate, but smoothly increase and decrease during the time $\tau \simeq r/\gamma v$. It follows that the

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wall impedance $Z(\omega)$ is exponentially depressed at frequencies $\omega \geq \gamma v/b$, where b is the aperture radius. The trivial space charge impedance of the pure Coulomb force is the only remaining at these frequencies, but it does not lead to instabilities (taking apart the structure resonances [4].) This high frequency impedance damping depends partly on the mechanism of the beam - wall interaction. However the common exponential factor proportional to the incident Poynting vector damping near the walls $f \propto \exp(-2kb/\gamma)$ is dominating. Below the problem is discussed for resistive-wall and structure impedances, the thresholds and the increments of the correspondent instabilities are estimated. The numerical examples are usually presented for parameters of the proposed CRYSTAL ring [3].

2 Longitudinal Oscillations

2.1 Resistive-Wall Impedance

Assuming the field dependence on the longitudinal coordinate and time as $e^{ik(z-vt)}$, Maxwell's equations reduce to the Poisson equation for a longitudinal electric field E_z excited by a charge linear density perturbation ρ :

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) - \frac{k^2}{\gamma^2} E_z = \frac{4\pi i k}{\gamma^2} \rho, \quad \rho = \rho_0 \delta(x) \delta(y) = \frac{\rho_0}{2\pi r} \delta(r), \quad (1)$$

where ρ_0 is an arbitrary constant. All the other field components can be expressed in terms of E_z :

$$B_\theta = \frac{v}{c} E_r = \frac{\gamma^2 v}{i c k} \frac{\partial E_z}{\partial r}, \quad B_r = B_z = E_\theta = 0.$$

Applying Leontovich boundary condition at the wall surface, $r = b$ [5],

$$E_z(b) = \left(\frac{-i k v}{4\pi \sigma} \right)^{1/2} B_\theta,$$

the electric field E_z in a perfectly conducting tube and its perturbation \tilde{E}_z due to a finite walls conductivity σ [6] can be found:

$$E_z = - \frac{2 i k}{I_0(\kappa) \gamma^2} \left[K_0 \left(\kappa \frac{r}{b} \right) I_0(\kappa) - K_0(\kappa) I_0 \left(\kappa \frac{r}{b} \right) \right] \quad (2)$$

$$\tilde{E}_z = \frac{2 v}{b c} \left(\frac{-i k v}{4\pi \sigma} \right)^{1/2} f(\kappa), \quad \kappa = \frac{k b}{\gamma}$$

Here a high frequency damping factor $f(\kappa)$ has been introduced:

$$f(\kappa) = \kappa \frac{K_1(\kappa) I_0(\kappa) - K_0(\kappa) I_1(\kappa)}{I_0^2(\kappa)} \quad (3)$$

$$\begin{aligned} f(\kappa) &\approx 1, \text{ for } \kappa \ll 1 \\ f(\kappa) &\approx \pi e^{-2\kappa}, \text{ for } \kappa \gg 1 \end{aligned}$$

$K_m(\kappa)$, $I_m(\kappa)$ are modified Bessel functions. A plot of this factor is shown in Fig.1.

Taking into account that the fields E_z , \tilde{E}_z and the current perturbation $J = \rho_0 v$ are connected by corresponding impedances, $E_z C = -Z^\parallel J$, $\tilde{E}_z C = -\tilde{Z}^\parallel J$, where C is the ring circumference, the results (Eq.2) can be expressed in terms of the impedances:

$$\begin{aligned}\frac{Z^\parallel(kv)}{C} &= \frac{2ikL}{\gamma^2 v} \\ \frac{\tilde{Z}^\parallel(kv)}{C} &= \frac{1 - i \operatorname{sign}(k)}{2\pi b \delta \sigma} f(\kappa) \equiv \frac{\tilde{Z}_0^\parallel}{C} (1 - i \operatorname{sign}(k)) \kappa^{1/2} f(\kappa),\end{aligned}\tag{4}$$

where $\operatorname{sign}(k)$ is the sign (+ or - of k), $\delta = c/\sqrt{2\pi\sigma kv}$ is the skin depth, $L = \ln(r_{\max}/a) + 1/2$ is the logarithmic factor with $r_{\max} = \min(b, 1/k)$, a is the beam radius, and

$$\frac{\tilde{Z}_0^\parallel}{C} = \frac{1}{2\pi b \delta \sigma} = \frac{1}{bc} \sqrt{\frac{\gamma v}{2\pi\sigma b}}$$

The plot of the longitudinal resistive wall impedance is shown in the Fig.(2).

The real part of $\tilde{Z}^\parallel(\kappa)$ achieves its maximum at the dimensionless wavenumber $\kappa = 0.43$,

$$\operatorname{Re} \tilde{Z}_{\max}^\parallel = 0.48 \tilde{Z}_0^\parallel = 0.24 Z_0 \frac{R}{b} \sqrt{\frac{\gamma v}{2\pi\sigma b}}$$

$Z_0 = 4\pi/c = 377 \Omega$. Assuming $\beta = 0.4$, $\sigma = 1.3 \cdot 10^{16}$, $R = 10 \text{ m}$, $b = 5 \text{ cm}$, it gives $\operatorname{Re} \tilde{Z}_{\max}^\parallel = 3 \Omega$.

2.2 Structure Impedance

The resistive wall impedance \tilde{Z}^\parallel (Eq.4) can be represented in terms of its ultrarelativistic value \tilde{Z}_{ur}^\parallel (which could be seen, e. g. in Ref.[2]) and the damping factor $f(\kappa)$ (Eq.3) :

$$\tilde{Z}^\parallel = \tilde{Z}_{ur}^\parallel f(\kappa)\tag{5}$$

The factor $f(\kappa)$ reflects the strong decrease of electrostatic fields of the beam density perturbation with the distance r as $\sim \exp(-kr/\gamma)$. The signal from the beam, damped near the walls as $f^{1/2} \sim \exp(-kb/\gamma)$, produce the perturbation of the surface current density with the same wavenumber. The energy loss which is a product of the field \tilde{E}_z and the current induced, is damped quadratically, as $f^{1/2} \cdot f^{1/2} = f \simeq \pi \exp(-2kb/\gamma)$. The energy loss is proportional to the real part of the impedance [2], so the last one is damped in the same way.

The beam-wall interaction due to the vacuum chamber inhomogeneity can be considered in a similar way. The incident electric field of the beam, damped with the factor \sqrt{f} , turns here into a radiative field, the portion of the energy radiated is determined by the frequency of the incident wave and the wall geometry. The beam energy loss is equal to the incident electrostatic field energy captured by the structure. It follows that the beam energy loss, and the real part of the impedance with it, are damped, in the comparison with the ultrarelativistic case, as the incident electric field near the walls, squared, i. e. as $(\sqrt{f})^2 = f$. It is the same result as for the resistive-wall impedance (Eq.5). To be more precise, the dependence of a

depth of the field penetration into the walls structure on the beam velocity have to be taken account. In the ultrarelativistic case this depth g is determined by the wall geometry only (the depth of a cavity or an iris under the consideration; details can be found in [2]). On the contrary, when the damping is strong, $f \ll 1$, the incident field energy is concentrated in the depth $g_{eff} = \gamma/2k$, if $g_{eff} < g$. This property could be taken into account as an additional factor $\simeq (g/g_{eff} + 1)^{-1}$ in the expression of the impedance in terms of its ultrarelativistic value (Eq.5):

$$\tilde{Z}^{\parallel} \simeq \tilde{Z}_{ur}^{\parallel} \frac{1}{1 + g/g_{eff}} \frac{1 + \beta^2}{2} f(\kappa). \quad (6)$$

The factor $(1 + \beta^2)/2$ reflects the contribution of the magnetic field of the beam perturbation in the total energy loss. Without significant restrictions, the resonant circuit impedance model [2] can be used:

$$\tilde{Z}_{ur}^{\parallel}(\omega) = \frac{R_s}{1 + iQ \left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r} \right)} \quad (7)$$

where $R_s \simeq Z_0 Q / (2\pi)$ is the shunt impedance, Q is the quality factor.

The real part of the broad band impedance, $Q \simeq 1$, corresponds to the beam energy loss in the radiative modes of the vacuum chamber. It has non-zero value only above the low frequency cut-off:

$$\omega \geq 2.4c/b,$$

2.4 is the lowest root of the Bessel function J_0 . Taking into account that $\omega = kv$, the damping factor f near the lowest possible frequency (cutoff) occurs to be:

$$f \simeq \pi \exp \left(-\frac{4.8}{\gamma\beta} \right)$$

So the broad band structure impedance cannot play any role for insufficiently relativistic particles, even for the proton beam in the CRYSTAL ring with $\beta = 0.4$ the damping factor $f \simeq 2 \cdot 10^{-5}$, which makes the broad band impedance completely negligible.

Another possibility for energy loss and for instabilities is connected with the radiation in low frequency modes of some elements of the vacuum chamber, typically with $Q \simeq 10^3 - 10^4$. If the lowest eigenfrequency of such a cavity-like element ω_r is not large, $2\omega_r b / (\gamma\beta c) \simeq 1$, or even less, the damping factor f in Eq.(6) could be not so small, and the instability due to the impedance

$$\tilde{Z}^{\parallel} = R_s f \frac{1 + \beta^2}{2} \frac{1}{g/g_{eff} + 1} \simeq R_s \exp \left(-2 \frac{\omega_r b}{\gamma\beta c} \right), \quad (8)$$

can take place. To avoid this instability in the CRYSTAL, it is enough to screen the main cavity and to choose sufficiently small sizes b_r for the parasitic ones. As the eigenfrequency $\omega_r^{-1} \propto b_r \sqrt{\epsilon\mu}$, this restriction primary concerns ferrite and high-epsilon elements.

2.3 Longitudinal Stability

The problem of a coherent stability for a cooled coasting beam was considered in Ref.[7], where the dispersion equation was derived for arbitrary relations between the eigenfrequencies, cooling rate and Landau damping; the equation is general and not so easy to analyze. Below, a perturbative approach is suggested to the problem.

Actually, the main part of the impedance Z^\parallel (Eq.4) is pure imaginary and therefore does not drive an instability. The instabilities could be driven by relatively small real parts of impedances $\text{Re}\tilde{Z}^\parallel$ introduced by different elements of the vacuum chamber. Coherent increments caused by them are small in the comparison with eigenfrequencies, mainly determined by the large space charge impedance Z^\parallel . Therefore, even a small Landau damping or small cooling rate is already enough to stabilize the oscillations. It follows that without significant restrictions the problem of a longitudinal stability for a coasting nonrelativistic beam can be considered perturbatively, with the real part of the impedance, the Landau damping and the cooling rate as small values. The small Landau damping means that a thermal dispersion of velocities Δw is much less than a phase velocity of longitudinal waves u_\parallel , i.e. the beam temperature is also a small parameter.

The dispersion relation can be found from the kinetic equation, which is the Vlasov equation plus cooling-diffusion Fokker-Planck term [7]. In the reference frame:

$$\frac{\partial f}{\partial t} + w \frac{\partial f}{\partial z} + \frac{Z_i e}{M_\parallel} E \frac{\partial f_0}{\partial w} = \frac{\partial}{\partial w} \left(\lambda_\parallel w f + d_\parallel \frac{\partial f}{\partial w} \right), \quad (9)$$

where f_0 is the beam phase density, f is its perturbation, w is a deviation of the particle velocity from the beam velocity v , $M_\parallel = M_i(1/\gamma^2 - 1/\gamma_t^2)^{-1}$ is a longitudinal mass of an ion of the beam, $M_i = A_i M_p$ is its mass, A_i and Z_i are the mass and charge numbers, λ_\parallel is the cooling rate and d_\parallel is the diffusion coefficient. The diffusion coefficient can be expressed in terms of a velocity dispersion Δw_* , corresponding to the thermal equilibrium with the cooler: $d_\parallel = \lambda_\parallel \Delta w_*^2$, $\Delta w_* \leq \Delta w$. It means that the diffusion term must be neglected in this perturbative approach because of its proportionality to the both small values – the cooling rate and the temperature.

After the Fourier transformation, substituting $\frac{\partial}{\partial t} + w \frac{\partial}{\partial z}$ on $-i(\Omega - kw)$, this leads to:

$$f = -\frac{Z_i e E}{M_\parallel} \frac{i}{\Omega - kw} \frac{\partial f_0}{\partial w} + \frac{i}{\Omega - kw} \frac{\partial}{\partial w} (\lambda_\parallel w f).$$

The electric field E and the phase density perturbation f are connected by means of the impedance:

$$EC = -Z_i e (Z^\parallel + \tilde{Z}^\parallel) \rho v, \quad \rho = \int f dw,$$

At the first approximation, neglecting the impedance perturbation, the width of the velocity distribution and the cooling term, the sound-like dispersion relation is found:

$$\Omega = \pm k u_\parallel, \quad u_\parallel = c \sqrt{\frac{2 N r_0 L}{C} \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} \right)}, \quad r_0 = \frac{Z_i^2 e^2}{M_i c^2}, \quad (10)$$

where N is the number of ions.

In the second approximation, the solution of the first one, $f = -\frac{Z_i e E}{M_{\parallel}} \frac{i}{\Omega} \frac{\partial f_0}{\partial w}$ can be substituted in the small cooling term, which gives the following result [8]:

$$\Omega(k) = \pm k u_{\parallel} \left\{ 1 + \frac{\pi i}{2} \text{sign}(k) u_{\parallel}^2 f_0'(\Omega/k) - i \frac{\tilde{Z}^{\parallel}}{2|Z^{\parallel}|} \right\} - i \frac{\lambda_{\parallel}}{2}. \quad (11)$$

This result is valid when the influence of Landau damping, longitudinal cooling and the perturbative impedance on the coherent spectrum is small.

According to Eq.(11), instabilities caused by the real part of the impedance $\text{Re}\tilde{Z}^{\parallel}$ can be avoided due to Landau damping or due to cooling. In the first case longitudinal temperature of the beam must be sufficiently high. Assuming the distribution function to be Gaussian, $f_0(u) = (2\pi\Delta w^2)^{-1/2} \exp(-u^2/2\Delta w^2)$, the stability condition can be expressed as:

$$\frac{u_{\parallel}^2}{\Delta w^2} \leq 2 \ln \left(\sqrt{\frac{\pi}{2}} \frac{u_{\parallel}^3}{\Delta w^3} \frac{|Z^{\parallel}|}{\text{Re}\tilde{Z}^{\parallel}} \right) \quad (12)$$

The factor $|Z^{\parallel}|/\text{Re}\tilde{Z}^{\parallel}$ is usually large. To see how much it is for a resistive wall impedance, calculations can be done for some typical parameters of a storage ring. For the smallest wavelength, $k = 1/R$, where $R = C/2\pi$:

$$\frac{|Z^{\parallel}|}{\text{Re}\tilde{Z}^{\parallel}} = \frac{2L}{\beta^{3/2}} \sqrt{\frac{2\pi\sigma b}{c}} \frac{b}{R}$$

Assuming $b = 5$ cm, $R = 10$ m, $\sigma = 1.3 \cdot 10^{16} \text{ s}^{-1}$, $\beta = 0.4$, $L = 5$, one has $|Z^{\parallel}|/\text{Re}\tilde{Z}^{\parallel} = 3 \cdot 10^4$. For such large values the stability condition (Eq.12) can be presented as:

$$\frac{\Delta w^2}{u_{\parallel}^2} \geq 0.03 f_Z, \quad (13)$$

where the factor f_Z has the weak logarithmic dependence on the impedance \tilde{Z}^{\parallel} . For $|Z^{\parallel}|/\text{Re}\tilde{Z}^{\parallel} = 3 \cdot 10^4$ $f_Z = 1$, for $|Z^{\parallel}|/\text{Re}\tilde{Z}^{\parallel} = 30$ it is only twice more. It means that the stability condition (13) is almost independent on the impedance \tilde{Z}^{\parallel} . The condition of the stabilization by means of the Landau damping (13) can be presented also in terms of the longitudinal temperature $T^{\parallel} = M_i \Delta w^2$:

$$T^{\parallel} \geq T_{th}^{\parallel}(\text{K}) = 0.02 \frac{I(\mu\text{A}) Z_i L}{\beta(1 - \gamma^2/\gamma_i^2)} f_Z \quad (14)$$

where the factor f_Z reflects the weak logarithmic dependence of the threshold on the ratio $|Z^{\parallel}|/\text{Re}\tilde{Z}^{\parallel}$, in the case of (13) $f_Z = 1$. For instance, longitudinal oscillations in a $1 \mu\text{A}$ beam of Li^{+1} [3] with $\beta = 0.06$, $L = 4$, will be Landau-damped if its longitudinal temperature $T^{\parallel} \geq T_{th}^{\parallel} = 1.6$ K.

For 1 mA beam of C_{12}^{+6} at TSR with $\beta = 0.041$ the threshold temperature calculated from (14) with $L = 5$, $f_Z = 1.5$ is: $T_{th}^{\parallel} = 2 \cdot 10^4$ K, which is rather close to the experimental value of $T^{\parallel} = 3 \cdot 10^4$ K.

This limit can be overcome by means of a cooling which introduce its own decrement in the coherent oscillations (Eq. 11). If the condition

$$\lambda_{\parallel} > 2\Lambda_k = k u_{\parallel} \frac{\text{Re}\tilde{Z}^{\parallel}}{|Z^{\parallel}|} = \frac{u_{\parallel}}{R} \frac{\beta \text{Re}\tilde{Z}^{\parallel}}{L Z_0} \quad (15)$$

is satisfied for all the wavenumbers k , the oscillations are stable even without Landau damping.

The increment Λ achieves its maximum together with the impedance $\text{Re}\tilde{Z}^{\parallel}$. For the resistive-wall impedance it is equal to:

$$\Lambda_{max} = 0.12 \frac{\gamma^{5/2} \beta^{3/2}}{L} \frac{u_{\parallel}}{b} \sqrt{\frac{c}{2\pi\sigma b}}. \quad (16)$$

Applying this formula to the mentioned Li_7^{+1} beam, the result follows:

$$\Lambda_{max} = (0.5\text{hour})^{-1},$$

which is much less than usual cooling rates. For a given λ_{\parallel} the restriction (15) can be treated as a safe condition imposed on the impedance \tilde{Z}^{\parallel} . Assuming $\lambda_{\parallel} = 10 \text{ s}^{-1}$, for the Li_7^{+1} beam it gives: $\text{Re}\tilde{Z}^{\parallel} < 10 \text{ K}\Omega$.

3 Transverse Oscillations

The increment of transverse oscillations of a coasting beam, equal to (see, e. g. Ref.[2]):

$$\Lambda_{\perp} = -\frac{Nr_0 c^2}{4\pi\gamma Q_b} \frac{\text{Re}\tilde{Z}^{\perp}(\omega)}{C} \quad (17)$$

achieves the maximum $\Lambda_{\perp m}$ at the same wave number n as the impedance. The maximum of a transverse resistive wall impedance $\tilde{Z}^{\perp}(\omega) = \frac{2v}{\omega b^2} \tilde{Z}^{\parallel}(\omega) \propto (\omega)^{-1/2}$ is at the lowest positive value of its argument $\omega = \omega_0(n - Q_b)$, where Q_b is a betatron tune. Thus, the resistive wall gives:

$$\Lambda_{\perp m} = -\frac{Nr_0 \beta c^2}{2\pi\gamma Q_b b^3} \frac{\text{sign}(\Delta_b)}{\sqrt{2\pi\sigma\omega_0|\Delta_b|}} \quad (18)$$

where Δ_b is a fractional part of the betatron tune. As for the longitudinal case, the transverse increment due to the walls resistivity occurs to be too small to be taken into account.

The structure transverse impedance is damped more than the longitudinal one because of the higher value of the cut-off frequency for unsymmetrical modes. The damping factor for dipole oscillations $f_1 \simeq \pi \exp(-7.6/\beta\gamma)$, $7.6 = 3.8 \cdot 2$ is twice the first root of the Bessel function J_1 .

The stability condition for transverse oscillations is similar to the longitudinal:

$$\lambda_{\perp} > 2\Lambda_{\perp},$$

λ_{\perp} is the transverse cooling rate, Λ_{\perp} is given by Eq. (17). For the mentioned Li beam, with $\lambda_{\perp} = 10 \text{ s}^{-1}$ it gives the restriction for the impedance: $\text{Re}\tilde{Z}^{\perp} < 1 \text{ M}\Omega/m$.

4 Conclusions

The concept of the storage ring impedance, developed mainly for ultrarelativistic beams cannot be at once applied to low and moderate energy cases. The reason is

that the causality principle for wake fields generally is not valid here, which cause an exponential damping of impedances above a certain threshold. The approach suggested above gives the possibility to find the wall impedance for an arbitrary energy if its ultrarelativistic value is known. An application of these results to the problem of crystallization shows that due to small currents and low energies the broad-band wall impedance is too weak to cause instabilities. On the contrary, a narrow-band impedance could be dangerous if its eigenfrequency ω_r is rather low: $2\omega_r b/(\gamma v) \leq 1$. However, due to the strong exponential damping of the narrow-band impedance with the eigenfrequency (Eq.8), the impedance can be sufficiently suppressed.

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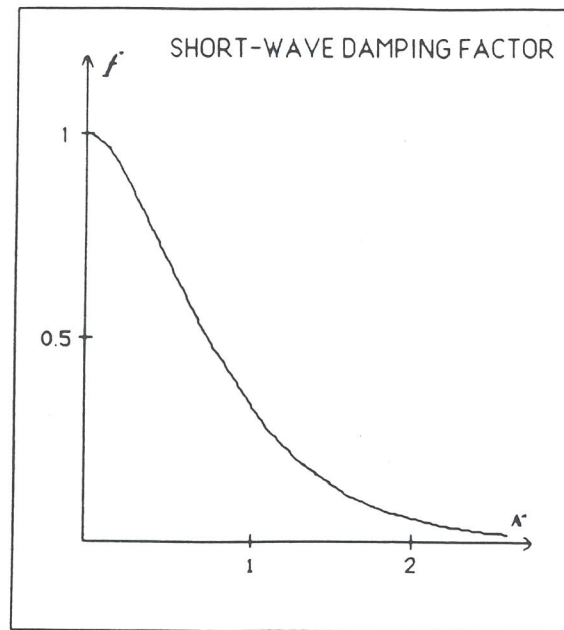


Figure 1: Short waves damping factor.

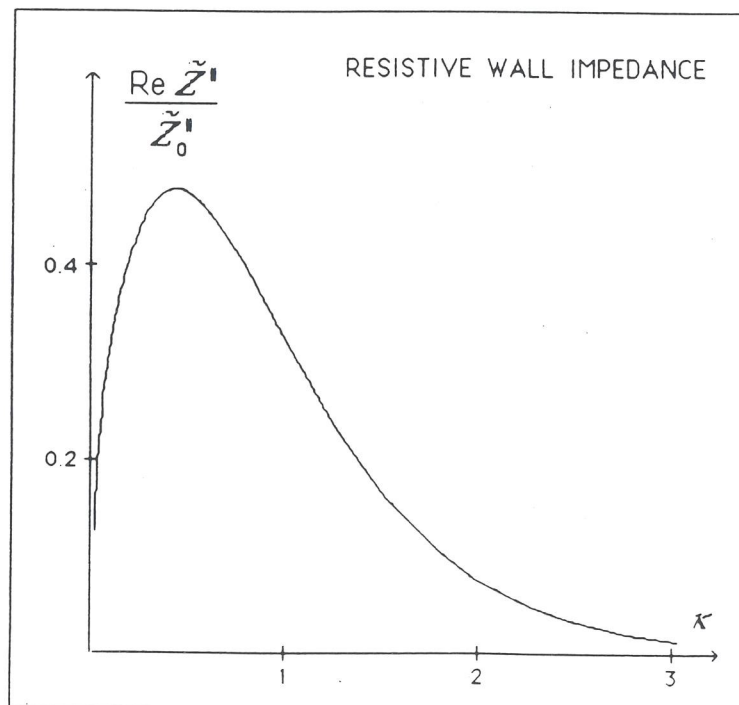


Figure 2: Longitudinal resistive-wall impedance.