D5-Branes and Quantum Impurity Models

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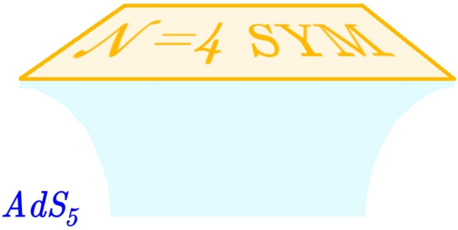
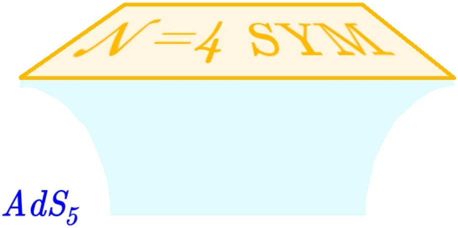
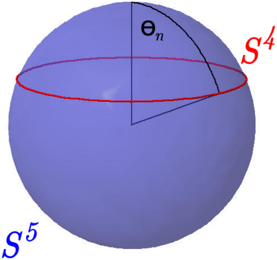
W.M. arXiv:1012.1973 and work in progress
A. Faraggi, W.M., L. A. Pando Zayas arXiv:1112.5028



Classical SUGRA regime

$$N \to \infty\,, \qquad \text{large } \lambda$$

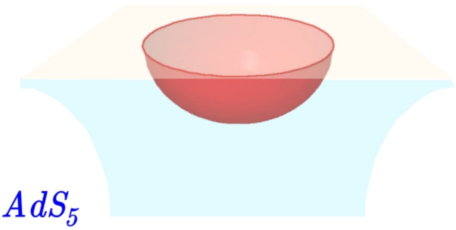


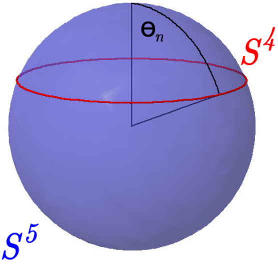


Latitude angle θ_n is quantized by the fundamental string charge

$$n = \frac{N}{\pi}(\theta_n - \sin\theta_n \cos\theta_n) \qquad 0 \leq n \leq N\,, \quad \frac{n}{N} \text{ fixed}$$

[Pawelczyk, Rey hep-th/0007154; Camino, Paredes, Ramallo hep-th/0104082]





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2-d part of the worldvolume behaves like a string worldsheet

$$\text{effective string tension} \qquad T_{eff} = \frac{N}{3\pi^2\alpha'} \sin^3\theta_n$$

[Hartnoll hep-th/0606178]

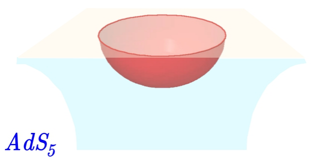
AdS/CFT dictionary

$\mathrm{AdS}_5 \times S^5$	$\leftrightarrow$	$\mathcal{N} = 4$ SYM
D5-brane		Wilson loop of operator in anti-symmetric rep. of $SU(N)$
fundamental string charge n		$\Gamma_n = \left.\begin{array}{c}\square\\ \square\\ \vdots\\ \square\end{array}\right\} n$
$\mathrm{e}^{-I_{\text{D5,on-shell}}}$		$\left\langle \mathrm{Tr}\,\mathcal{P} \exp \oint ds\,(iA_\mu \dot{x}^\mu + \phi\lvert\dot{x}\rvert)\right\rangle$

1) Hermitian matrix model

The calculation of 1/2 BPS Wilson loops is reduced to a hermitian matrix model by localization.

[Erickson, Semenoff, Zarembo hep-th/0003055; Drukker, Gross hep-th/0010274; Pestun, arXiv:0712.2824]



$$\left\langle \mathrm{Tr}\,\mathcal{P}\exp\oint ds\,(iA_\mu \dot{x}^\mu + \phi|\dot{x}|)\right\rangle = \frac{1}{Z}\int dM\ \mathrm{Tr}_{\Gamma_n}[\mathrm{e}^M]\exp\left(-\frac{2N}{\lambda}\mathrm{Tr}[M^2]\right)$$

The result from the matrix model agrees perfectly with the D5-brane calculation

$$I_{\text{on-shell}} = -\frac{2N\sqrt{\lambda}}{3\pi}\sin^3\theta_n$$

[Yamaguchi hep-th/0603208; Okuyama, Semenoff hep-th/0604209; Hartnoll, Kumar hep-th/0605027]

2) Fermions coupled to SYM fields

Operators in anti-symmetric representation Γ_n are described by N fermions coupled to the SYM fields and localized at the intersection of the D5-brane with the boundary. The total occupation number is fixed to n.

$$I = I_{\mathcal{N}=4} + \int dt \left[i\chi^\dagger \partial_t \chi + \chi^\dagger (A_0 + \phi)\chi + \mu(\chi^\dagger \chi - n) \right]$$

[Gomis, Passerini hep-th/0604007]

Relation to quantum impurity models

- The above is a Kondo-like quantum impurity model.

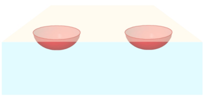
  [Sachdev arXiv:1006.3794, 1010.0682]

- similarities with overscreened multi-channel Kondo model

  [Parcollet, Georges, Kotliar, Sengupta PRB 58, 3794 (1998)]

- hints at a holographic dual of the physics of quantum antiferromagnets and strange metals

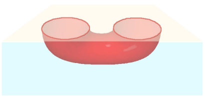
Two Wilson loops



on-shell action is independent of distance between loops

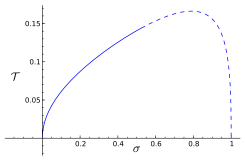
$$I = 2I_{WL} = -\frac{4N\sqrt{\lambda}}{3\pi}\sin^3\theta_n$$

Wilson loop correlator

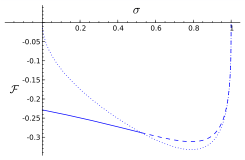


shape depends on order parameter σ

dimensionless "temperature" $T = \frac{x}{2\pi R}$

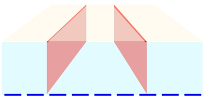


free energy



[Zarembo hep-th/9904149] for fundamental representation

Two Polyakov loops

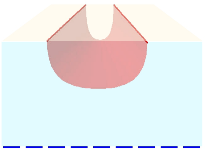


finite temperature, D5-branes end at horizon
action is independent of the distance between loops

$$I = 2I_{PL} = -\frac{2N\sqrt{\lambda}}{3\pi}\sin^3\theta_n$$

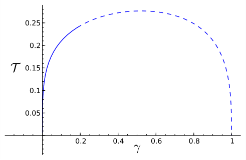
note: $I_{PL} = \frac{1}{2}I_{WL}$

Holographic dimer

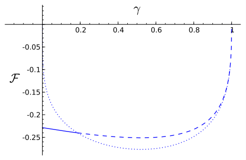


shape depends on order parameter γ

dimensionless "temperature" $T = \frac{X}{\beta}$



free energy



[Kachru, Karch, Yaida arXiv:0909.2639], older work on $q-\bar{q}$ potential

In this seminar

Part I: Quantum impurity model

- calculation of Wilson and Polyakov loops using the formulation as a quantum impurity model
  $\Rightarrow$ thermodynamic picture, impurity entropy
- result agrees with matrix model and D5-brane calculations
- extend AdS/CFT dictionary

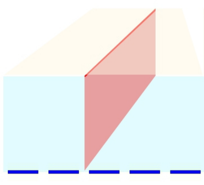
Part II: Holographic Wilson loop correlators

- review of D5-brane solution connecting two circles on the boundary
- thermodynamics of Wilson loop correlators and phase transition
- towards a quantum impurity model of the Wilson loop correlator

Part I: Quantum impurity model

[W.M. arXiv:1012.1973]

D5-brane description of Polyakov loop on $\mathbb{R}^3 \times \mathbb{R}$



- finite temperature (black brane gravity background)

$$\beta = \frac{\pi l^2}{r_+}$$

- renormalized on-shell action

$$I_{D5} = -\beta F = -\frac{N\sqrt{\lambda}}{3\pi^2 l^2}\beta r_+ \sin^3\theta_n$$

- strong coupling entropy enhancement

$$S = \frac{N\sqrt{\lambda}}{3\pi}\sin^3\theta_n \qquad \nu = \frac{n}{N} = \frac{1}{\pi}(\theta_n - \sin\theta_n\cos\theta_n)$$

- in contrast, weak coupling degeneracy of states is

$$\ln d_n = -N[\nu\ln\nu + (1-\nu)\ln(1-\nu)]$$

$\mathcal{N}=4$ SYM with fermionic impurity

$$I=\int d^3x\,d\tau\,\mathcal{L}_{\mathcal{N}=4}+\int d\tau\,\left[\chi_a^\dagger\partial_\tau\chi^a+i\chi_a^\dagger\tilde{A}^a_b\chi^b+i\mu(\chi_a^\dagger\chi^a-n)\right]$$

- χ : N-component spinor transforming in fund. rep. of $SU(N)$
- $\tilde{A}$: $\tilde{A}=A_\mu\dot{x}^\mu+n^I\phi_I|\dot{x}|$, $\tilde{A}^a_b=(t^c)^a{}_b\tilde{A}_c$
- μ : Lagrange multiplier, fixes fermion occupation number to n
- τ : Euclidean time, $\tau\in(0,\beta)$

$\mathcal{N} = 4$ SYM with fermionic impurity

$$I = \int d^3x\, d\tau\, \mathcal{L}_{\mathcal{N}=4} + \int d\tau\, \left[\chi_a^\dagger \partial_\tau \chi^a + i\chi_a^\dagger \tilde{A}^a_b \chi^b + i\mu(\chi_a^\dagger \chi^a - n)\right]$$

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- μ : Lagrange multiplier, fixes fermion occupation number to n
- τ : Euclidean time, $\tau \in (0, \beta)$

Integrate out SYM fields → impurity action

$$I = \int d\tau [\chi_a^\dagger (\partial_\tau + i\mu)\chi^a - i\mu n] + \frac{\lambda}{2N} \int d\tau d\tau' D(\tau - \tau')\chi_a^\dagger(\tau)\chi_b^\dagger(\tau')\chi^b(\tau)\chi^a(\tau')$$

with SYM background correlator $\langle \tilde{A}_c(\tau)\tilde{A}_{c'}(\tau')\rangle = \frac{2\lambda}{N}\delta_{cc'} D(\tau - \tau')$

Large-N limit is dominated by saddle point

$$G(i\omega_n) = \frac{1}{i\omega_n - \bar{\mu} - \Sigma(i\omega_n)} \qquad \text{Dyson's equation } (\bar{\mu} = i\mu)$$

$$\Sigma(\tau) = \lambda D(\tau) G(\tau) \qquad \text{self-energy}$$

$$G(\tau \to 0^-) = \frac{n}{N} = \nu \qquad \text{occupation number constraint}$$

2-point Green's function

$$\left\langle \mathcal{T} \chi^a(\tau) \chi_b^\dagger(0) \right\rangle = -G(\tau) \delta_b^a$$

Fourier transform

$$G(i\omega_n) = \int_0^\beta d\tau \, \mathrm{e}^{i\omega_n \tau} \, G(\tau)$$

$\omega_n = \frac{2\pi}{\beta}(n + \frac{1}{2})$: fermionic Matsubara frequencies
analytic continuation $i\omega_n \to \omega + i0^+$ gives *retarded* Green's function

Scaling ansatz

$$D(\tau) = A_0 \beta^{-2\Delta_0} \left[\frac{\pi}{\sin(\pi\tilde{\tau})} \right]^{2\Delta_0} \qquad G(\tau) = -A\beta^{-2\Delta}\, \mathrm{e}^{\alpha\tilde{\tau}} \left[\frac{\pi}{\sin(\pi\tilde{\tau})} \right]^{2\Delta}$$

$\tilde{\tau}$: $\tilde{\tau} = \tau/\beta \in (0,1)$

Δ, Δ_0 : scaling dimensions

A, A_0 : constants

α : particle-hole asymmetry parameter

- The half-filling case $\nu = \frac{1}{2}$ ($\alpha = 0$) was solved by Sachdev [arXiv:1010.0682].
- In the scaling regime, Green's function is dominated by the self-energy Σ.

Properties of the solution

$$2\Delta + \Delta_0 = 1 \qquad \nu = \frac{1}{2} - \frac{\vartheta}{\pi} - \frac{\Delta_0}{2\sin(\pi\Delta_0)}\sin(2\vartheta)$$

ϑ : spectral asymmetry angle $\quad \mathrm{e}^{\alpha} = \frac{\sin(\pi\Delta-\vartheta)}{\sin(\pi\Delta+\vartheta)} \quad \vartheta \in (-\pi\Delta, \pi\Delta)$

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Impurity model

$$\vartheta \in (-\pi\Delta, \pi\Delta)$$

$$\nu = \frac{1}{2} - \frac{\vartheta}{\pi} - \frac{\Delta_0}{2\sin(\pi\Delta_0)}\sin(2\vartheta)$$

D5-brane

$$\theta_n \in (0, \pi)$$

$$\nu = \frac{1}{\pi}(\theta_n - \sin\theta_n \cos\theta_n)$$

Properties of the solution

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Impurity model

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D5-brane

$$\theta_n \in (0, \pi)$$

$$\nu = \frac{1}{\pi}(\theta_n - \sin\theta_n \cos\theta_n)$$

Relation between impurity model and D5-brane

The D5-brane corresponds to the limiting case

$$\Delta_0 = 0 \qquad \Delta = \frac{1}{2} \qquad \vartheta = \frac{\pi}{2} - \theta_n$$

Background Green's funtion

$$D(\tau) = \frac{c^2}{\beta^2} \qquad \text{for dimensional reasons}$$

Self-energy equation is now trivial

$$\Sigma(\tau) = \lambda D(\tau) G(\tau) = \hat{\lambda} G(\tau) \qquad \hat{\lambda} = \frac{c^2 \lambda}{\beta^2}$$

Rescale to remove explicit β and λ from saddle point equations

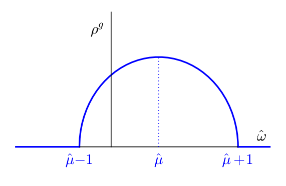
$$\omega = 2\sqrt{\hat{\lambda}}\hat{\omega} \qquad \bar{\mu} = 2\sqrt{\hat{\lambda}}\hat{\mu} \qquad G(\omega) = \frac{g(\hat{\omega})}{2\sqrt{\hat{\lambda}}}$$

Solution for retarded Green's function

$$g(\hat{\omega}) = 2\left\{(\hat{\omega} - \hat{\mu}) - \left[(\hat{\omega} - \hat{\mu})^2 - 1\right]^{1/2}\right\}$$

Spectral density is a Wigner semi-circle

$$\rho^g(\hat\omega) = \begin{cases} 4\sqrt{1-(\hat\omega-\hat\mu)^2} & \text{for } |\hat\omega-\hat\mu| < 1, \\ 0 & \text{otherwise.} \end{cases}$$



Occupation number

$$\nu - \frac{1}{2} = -\frac{1}{4\pi}\int d\hat\omega\, \rho^g(\hat\omega) \tanh\left(c\sqrt{\lambda}\hat\omega\right)$$

for large λ (SUGRA regime)

$$\nu - \frac{1}{2} = -\frac{1}{\pi}\left(\arcsin\hat\mu + \hat\mu\sqrt{1-\hat\mu^2}\right) \qquad \hat\mu \in (-1,1)$$

Spectral density is a Wigner semi-circle

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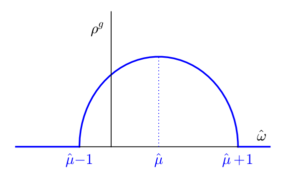

Occupation number

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for large λ (SUGRA regime)

$$\nu - \frac{1}{2} = -\frac{1}{\pi}\left(\arcsin\hat\mu + \hat\mu\sqrt{1-\hat\mu^2}\right) \qquad \hat\mu \in (-1,1)$$

Comparison with D5-brane

$$\hat\mu = \cos\theta_n \qquad \Rightarrow \qquad \nu = \frac{1}{\pi}(\theta_n - \sin\theta_n\cos\theta_n)$$

Impurity entropy

$$\frac{\partial S}{\partial \nu} = -N \frac{\partial \bar{\mu}}{\partial T}$$

yields

$$S = c\,\frac{4N\sqrt{\lambda}}{3\pi}\left(1-\hat{\mu}^2\right)^{3/2} = c\,\frac{4N\sqrt{\lambda}}{3\pi}\sin^3\theta_n$$

$\sqrt{\lambda}$ entropy enhancement comes from temperature dependence of $\bar{\mu}$

Comparison with D5-brane for Polyakov loop

$$S_{\rm PL} = \frac{N\sqrt{\lambda}}{3\pi}\sin^3\theta_n \qquad \Rightarrow \qquad c_{\rm PL} = \frac{1}{4}$$

Impurity entropy

$$\frac{\partial S}{\partial \nu} = -N\frac{\partial \bar{\mu}}{\partial T}$$

yields

$$S = c\frac{4N\sqrt{\lambda}}{3\pi}\left(1-\hat{\mu}^2\right)^{3/2} = c\frac{4N\sqrt{\lambda}}{3\pi}\sin^3\theta_n$$

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Comparison with D5-brane for Polyakov loop

$$S_{\rm PL} = \frac{N\sqrt{\lambda}}{3\pi}\sin^3\theta_n \qquad \Rightarrow \qquad c_{\rm PL} = \frac{1}{4}$$

Comparison with D5-brane for Wilson loop

we know $\langle \tilde{A}(\tau)\tilde{A}(0)\rangle_{\rm WL}$ $\Rightarrow$ $c_{\rm WL} = \frac{1}{2}$

perfect agreement with $I_{\rm WL} = -\frac{2N\sqrt{\lambda}}{3\pi}\sin^3\theta_n$

Scaling regime

- found general solution in scaling regime with arbitrary filling fraction ν
- D5-brane configuration corresponds to the limiting case of scaling regime $\Delta_0 = 0$, i.e., constant $D(\tau)$
- θ_n is related to spectral asymmetry angle $\vartheta = \frac{\pi}{2} - \theta_n$

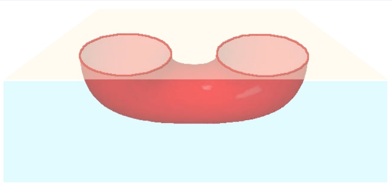
Limiting case

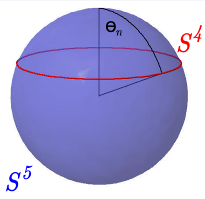
- $\Delta_0 = 0$ case can be solved exactly
- the solution agrees (even at finite λ) with the matrix model calculation by Hartnoll and Kumar [hep-th/0605027]
- $\lambda \to \infty$ limit reproduces D5-brane relations for Wilson and Polyakov loops
- $\log Z$ is interpreted as the impurity entropy
- $\sqrt{\lambda}$ entropy enhancement stems from temperature dependence of the chemical potential
- $\hat{\mu} = \cos \theta_n$ is the (rescaled) chemical potential

Part II: Wilson loop correlator

[work in progress]

D5-brane configuration





- 2d part in AdS_5 is string worldsheet with effective tension

$$T_{eff} = \frac{N}{3\pi^2\alpha'} \sin^3 \theta_n$$

- We can "recycle" old results on the Wilson loop correlator in the fundamental representation. [e.g., Zarembo hep-th/9904149; Olesen, Zarembo hep-th/0009210]

effective string world-sheet

$$ds^2_{AdS_5} = \frac{L^2}{z^2}\left(-dt^2 + dz^2 + dx^2 + dy^2 + y^2 d\phi^2\right)$$

world-sheet connects 2 parallel circles of radius R located at $x = \pm\frac{1}{2}|x_{12}|$

$$\phi = \tau\,, \qquad x = \rho\,, \qquad y = \sqrt{a^2 - \rho^2}\cos\theta(\rho)\,, \qquad z = \sqrt{a^2 - \rho^2}\sin\theta(\rho)$$

with

$$a^2 = R^2 + \frac{1}{4}|x_{12}|^2$$

and

$$\frac{d\theta}{d\rho} = -\operatorname{sgn}\rho\,\frac{a}{a^2 - \rho^2}\,\frac{\sqrt{\sin^4\theta_*\cos^2\theta - \cos^2\theta_*\sin^4\theta}}{\cos\theta_*\sin^2\theta}\,, \qquad \theta_* = \theta(0)$$

[Zarembo hep-th/9904149]

Remark: Configuration of 2 concentric circles considered in [Olesen, Zarembo hep-th/0009210] can be obtained by conformal transformations.

Recast the solution in terms of thermodynamic quantities (new!)

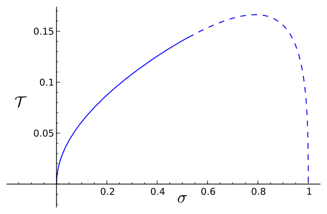
- distance x_{12} can be described in terms of a dimensionless "temperature"
- or: measure inverse temperature $\beta = 2\pi R$ in units of $|x_{12}|$

$$\mathcal{T} = \frac{|x_{12}|}{2\pi R} = \frac{1}{\pi} \sinh J$$

- J is obtained by integrating the ODE for θ and is given in terms of the order parameter $\sigma = \sin^2 \theta_*$

$$J = \frac{\pi}{4} \sqrt{\sigma(1-\sigma)}\, \mathrm{F}_1 \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 2; \sigma, \sigma - 1 \right)$$

F_1 is an Appell hypergeometric function of 2 variables



- There is a maximum temperature for which the correlator exists (or maximum distance, or minimum radius)
- 2 branches

Renormalized on-shell action gives the free energy

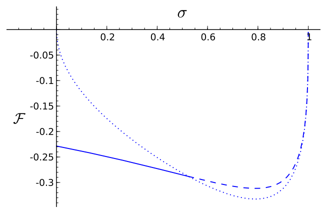
- factor out Wilson loop value and define dimensionless free energy $\mathcal{F}$

$$I_{\rm ren} = |I_\circ| \frac{\mathcal{F}}{\mathcal{T}}\,, \qquad I_\circ = -\frac{2N\sqrt{\lambda}}{3\pi} \sin^3\theta_n$$

- in terms of the order parameter

$$\frac{\mathcal{F}}{\mathcal{T}} = -\frac{\pi}{2\sqrt{\sigma(2-\sigma)}}\,\mathrm{F}\left(\frac{1}{2},\frac{1}{2};2;\frac{1}{2-\sigma}\right)$$

F is a (Gauss) hypergeometric function



First order phase transition between the correlator and 2 disconnected Wilson loops

$$\mathcal{F}_{\circ\circ} = -2\mathcal{T}$$

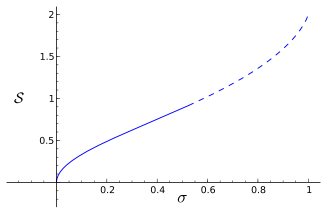
Entropy

$$\mathcal{S} = -\frac{\partial \mathcal{F}}{\partial \mathcal{T}} = -\frac{\mathcal{F}}{\mathcal{T}} - \tanh J \, \frac{d(\mathcal{F}/\mathcal{T})}{d\sigma} \left(\frac{dJ}{d\sigma} \right)^{-1}$$

calculation of the second term involves some “hypergeometric magic”

cf. [Benincasa, Ramallo arXiv:1204.6290]

$$\mathcal{S} = -\frac{\mathcal{F}}{\mathcal{T}} - \frac{2\sqrt{1-\sigma}}{\sigma} \tanh J$$



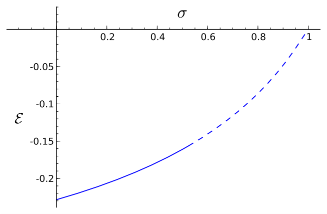
- unstable branch approaches the disconnected solution for $\sigma \longrightarrow 1$

$$\mathcal{S}_{\circ\circ} = 2$$

- bound state has less entropy

Energy

$$\mathcal{E} = \mathcal{F} + \mathcal{T}\,\mathcal{S} = -\frac{2\sqrt{1-\sigma}}{\pi\sigma}\tanh J \sinh J$$



Low energy of the bound state dominates the correlator.

Quantum 2-impurity model

After integrating out the SYM fields we are left with 2 coupled impurities

$$I = \sum_{I=1}^{2} \int d\tau [\chi^{\dagger}_{Ia}(\partial_\tau + i\mu_I)\chi^a_I - i\mu_I n_I]$$
$$+ \sum_{I,J=1}^{2} \frac{\lambda}{2N} \int d\tau d\tau' D_{IJ}(\tau - \tau')\chi^{\dagger}_{Ia}(\tau)\chi^{\dagger}_{Jb}(\tau')\chi^b_I(\tau)\chi^a_J(\tau')$$

background correlator is known from SYM correlation functions

$$D_{11}(\tau) = D_{22}(\tau) = \frac{1}{4\beta^2} = \frac{1}{16\pi^2 R^2}$$

$$D_{12}(\tau) = \frac{1}{4\beta^2}\left(1 - \frac{\frac{1}{2}(1 - \nu_I \cdot \nu_J) + X^2}{\sin^2[\pi(\tau + \tau_0)/\beta] + X^2}\right)$$

- $X = \frac{|x_{12}|}{2R}$, ν_I is 6-d unit-vector multiplying the SYM scalars in $\tilde{A} = A_\tau + \nu \cdot \phi$
- time has the same orientation on both loops $\Rightarrow D_{12}$ depends on $\tau - \tau'$
- arbitrary off-set τ_0 between the origins of time on the loops (D_{21} has $-\tau_0$)

Saddle point equations

- Green's function and self-energy are 2×2 matrices
- chemical potential matrix

$$\bar{\mu} = \begin{pmatrix} i\mu_1 & 0 \\ 0 & i\mu_2 \end{pmatrix}$$

- Dyson's equation

$$G(i\omega_n)^{-1} = i\omega_n \mathbb{I} - \bar{\mu} - \Sigma(i\omega_n)$$

- self-energy equation

$$\Sigma_{IJ}(\tau) = \lambda D_{IJ}(\tau) G_{IJ}(\tau) \qquad \text{(no summation over } I, J\text{)}$$

- occupation number constraints

$$G_{II}(\tau \to 0^-) = \frac{n_I}{N} = \nu_I$$

Disconnected solution

There is always a solution with vanishing off-diagonal elements

$$G_{12} = G_{21} = \Sigma_{12} = \Sigma_{21} = 0$$

G_{11} and G_{22} are solutions of the 1-impurity model.

This solution is dual to 2 separate Wilson loops.

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Are there other solutions?

- From holographic dual, expect 2 other solutions for $\mathcal{T} < \mathcal{T}_{max}$.
- Do they describe a BCS state (fermion condensate)?
- I have not found them yet, but there are indications that I am on the right track.

Rewrite self-energy equation

Form of D_{12} suggests to write self energy as

$$\Sigma(\tau) = \hat{\lambda}\left[G(\tau) + \Theta(\tau)\right] , \qquad \hat{\lambda} = \frac{\lambda}{4\beta^2}$$

with

$$\Theta_{11} = \Theta_{22} = 0 , \qquad G_{12}(\tau) = -\frac{\sin^2[\pi(\tau+\tau_0)/\beta] + X^2}{\frac{1}{2}(1 - v_1 \cdot v_2) + X^2}\,\Theta_{12}(\tau)$$

Fourier transform of self-energy equation

$$G_{12}(i\omega_n) = \frac{\frac{1}{4}\,\mathrm{e}^{2\pi i\frac{\tau_0}{\beta}}\,\Theta_{12}(i\omega_{n+1}) + \frac{1}{4}\,\mathrm{e}^{-2\pi i\frac{\tau_0}{\beta}}\,\Theta_{12}(i\omega_{n-1}) - \left(X^2 + \frac{1}{2}\right)\Theta_{12}(i\omega_n)}{\frac{1}{2}(1 - v_1 \cdot v_2) + X^2}$$

Equation for G_{21} has $-\tau_0$ instead of τ_0

Invariance of Dyson's equation

Dyson's equation is invariant under the transformation

$$G_{12}(i\omega_n) \to f(i\omega_n) G_{12}(i\omega_n) \,, \quad G_{21}(i\omega_n) \to [f(i\omega_n)]^{-1} G_{21}(i\omega_n)$$

$$\Theta_{12}(i\omega_n) \to f(i\omega_n) \Theta_{12}(i\omega_n) \,, \quad \Theta_{21}(i\omega_n) \to [f(i\omega_n)]^{-1} \Theta_{21}(i\omega_n)$$

for any function $f(i\omega_n)$

The self-energy equation is not invariant.

Eliminate the arbitrary τ_0

τ_0 can be absorbed by a transformation with

$$f(i\omega_n) = \mathrm{e}^{i\omega_n \delta} \qquad \Rightarrow \qquad \tau_0 \to \tau_0 + \delta$$

Self-energy equation is now

$$G_{12}(i\omega_n) = \frac{\frac{1}{4}\left[\Theta_{12}(i\omega_{n+1}) + \Theta_{12}(i\omega_{n-1}) - 2\Theta_{12}(i\omega_n)\right] - X^2 \Theta_{12}(i\omega_n)}{\frac{1}{2}(1 - v_1 \cdot v_2) + X^2}$$

Scaling limit?

pick

$$f(i\omega_n) = \mathrm{e}^{-\frac{\beta}{\pi} J\omega_n} \qquad \text{with} \quad \sinh J = X$$

The same J as in the holographic dual!

This eliminates the constant term in the self-energy equation (asymptotic behaviour of G_{12})

$$G_{12}(i\omega_n) = \frac{\frac{1}{2}}{\cosh(2J) - v_1 \cdot v_2} \{\cosh(2J)\left[\Theta_{12}(i\omega_{n+1}) + \Theta_{12}(i\omega_{n-1}) - 2\Theta_{12}(i\omega_n)\right] - \sinh(2J)\left[\Theta_{12}(i\omega_{n+1}) - \Theta_{12}(i\omega_{n-1})\right]\}$$

Combinations look like discretized second and first derivatives of Θ_{12}. Hint for a good scaling limit?

I remain hopeful.